Quantum Bayesian Nash Equilibria and Bell Inequalities

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WHY QUANTUM GAMES?

- Game Theory
  - "Newtonian mechanics" of mathematical social sciences
  - describes Dynamics of two (or more) autonomous agents
- Game theory with Hilbert vector
  - possible cure for difficulty of handling player correlations
  - possible framework for quantum information processing

Basic Elements

A \ B	0	1	
0	1	0	1-P
1	0	3	P
	1-Q	Q	

choice (probabilities)

- Alice and Bob in search of higher payoff $\Pi(P, Q)$
- Best Response to Best Response: Nash Equilibrium $(P^*, Q^*) = (0, 0), \quad \Pi^* = 1$
- Pareto Efficient N.E. $(P^*, Q^*) = (1, 1), \quad \Pi^* = 3$

ROCK-SCISSORS-PAPER GAME

A\ B	0	1	2	
0	0	- \ +	+ \ -	P_0
1	+ \ -	0	- \ +	P_1
2	- \ +	+ \ -	0	P_2
	Q_0	Q_1	Q_2	

- No dominant strategy
- No apparent Nash E.
- Random play is best for both

$$P_0^* = P_1^* = P_2^* = 1/3$$

$$\Pi^* = 0$$

: Mixed Nash Equilibrium

- Both just break even (Stop telling trivialities...)

Calculating Payoffs

M_{AB}	0	1	2
0	M_{00}	M_{01}	M_{02}
1	M_{10}	M_{11}	M_{12}
2	M_{20}	M_{21}	M_{22}

P_{AB}	0	1	2
0	P_0Q_0	P_0Q_1	P_0Q_2
1	P_1Q_0	P_1Q_1	P_1Q_2
2	P_2Q_0	P_2Q_1	P_2Q_2

- Payoff Matrix

  M_{AB}

- Payoff is calculated as

$$\Pi = \sum_{A,B} P_{AB} M_{AB}$$

Joint Probability Matrix

$P_{AB} = P_A Q_B$

(Strategy)

Elements of Game Theory

- Payoff matrix (game table) $M_{AB} \quad L_{AB}$
- Joint probability (strategy) $P_{AB} = P_A Q_B$
- Payoff $\Pi_{Ai} = \sum_{A,B} P_{AB} M_{AB}$

$$\Pi_{Bl} = \sum_{A,B} P_{AB} L_{AB}$$

- Nash Equilibria (solutions)

$$\partial_P \Pi_{Ai}|_{(P^*,Q^*)} = 0$$
$$\partial_Q \Pi_{Bl}|_{(P^*,Q^*)} = 0$$

plus "edge solutions"

PD Game with Punisher

- A game with Incomplete information on player types

M_{AB}		b=0 90%		b=1 10%	
	A \ B	0	1	0	1
a=0 90%	0	1	5	-20	-25
	1	0	3	0	3
a=1 10%	0	-1	0	0	-5
	1	0	0	0	0

Undercover Punisher [Type 1]

Bayesian update of strategy based on type/strat assumption
-> Bayesian Nash Equilibrium (Harsanyi's Theory)

Multi-Sector Game

- Hrsanyi : Type a, b with mixtures $S^{[a]}, T^{[b]}$
- Payoff Matrices for Alice and Bob $M_{AB}^{[ab]},\ L_{AB}^{[ab]}$
- Joint strategy with Type Locality assumption

$$P_{AB}^{[ab]} = P_A^{[a]} Q_B^{[b]}$$

- Sector Payoffs

$$\Pi_{Ai}^{[ab]} = \sum_{A,B} P_{AB}^{[ab]} M_{AB}^{[ab]} \qquad \Pi_{Bl}^{[ab]} = \sum_{A,B} P_{AB}^{[ab]} L_{AB}^{[ab]}$$

- Total Payoffs $\Pi^{[ab]} = \sum_{a,b} S^{[a]} T^{[b]} \Pi^{[ab]}$

Defects of Current Theories

- Aesthetic: Ugly math with underlying probability vector and arbitrary matrix
- Technical: Hard to include "player correlation" by its construction
- Nanotechnological: Need to handle quantum devices: Spins

- Quantum strategy *aus* unitary vector ?!

$$|\Psi\rangle = U_\alpha U_\beta |\Phi\rangle \qquad P_{AB} = |\langle AB|\Psi\rangle|^2$$

Player Action & Probability

- Classical Strategy : Individual Probabilities

  P_A :Alice, Q_B:Bob $\qquad P_{AB} = P_A Q_B$

- Quantum Strategy : Individual Unitary Actions

  U_α:Alice, V_β:Bob $\qquad P_{AB} = |\langle AB| U_\alpha V_\beta |\Phi\rangle|^2$

  if entanglement present $\qquad \neq P_A Q_B$ in general

- When $|\Phi\rangle = |00\rangle$, back to Classical w. identifications

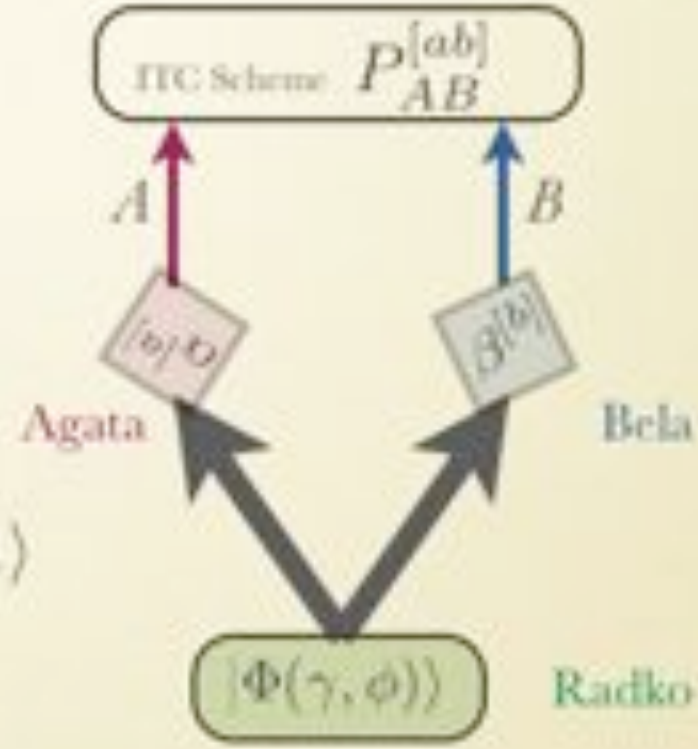
  $P_A = |\langle A| U_\alpha |0\rangle|^2$ and $Q_B = |\langle B| V_\beta |0\rangle|^2$

  ∵ Play Strategy P_A (Q_B) = Adjust 'angle' α (β)

Quantum Strategy

- Implementation
  1) Pre-game calibration with
     $|\Phi\rangle = |00\rangle$

  2) Game play with full
     entangled state
     $$|\Phi\rangle = \cos\frac{\gamma}{2}|00\rangle + e^{i\phi}\sin\frac{\gamma}{2}|11\rangle$$
     $$P_{AB} = |\langle AB|\, U_\alpha V_\beta\, |\Phi\rangle|^2$$



- Nonlocality:
  Action of Alice appears affected by action of Bob

Cereceda Game

- A version of Battle of Sexes Game

M\L		b=0 50%		b=1 50%	
	A \ B	0	1	0	1
a=0 50%	0	1 \ 3	0	-1 \ -3	0
	1	0	3 \ 1	0	-3 \ -1
a=1 50%	0	-1 \ -3	0	-3 \ -1	0
	1	0	-3 \ -1	0	-1 \ -3

Multisector Incomplete Information extension
Battle of Sexes / Chiken Game; look for Bayesian Nash Eq.

MULTISECTOR QUANTUM GAME

- Type $[a]$, $[b]$ with mixtures $S^{[a]}$, $T^{[b]}$
- Payoff Matrices for Alice and Bob $M_{AB}^{[ab]},\ L_{AB}^{[ab]}$
- Joint strategy with local actions U_α and V_β on $\Phi_{\gamma\phi}$ $P_{AB}^{[ab]} = |\langle AB|\, U_\alpha^{[a]} V_\beta^{[b]} \,|\Phi_{\gamma\phi}\rangle\rangle|^2$
- Sector Payoffs

$$\Pi_{Ai}^{[ab]} = \sum_{A,B} P_{AB}^{[ab]} M_{AB}^{[ab]} \qquad \Pi_{Bl}^{[ab]} = \sum_{A,B} P_{AB}^{[ab]} L_{AB}^{[ab]}$$

- Total Payoffs $\Pi^{[ab]} = \sum_{a,b} S^{[a]} T^{[b]} \Pi^{[ab]}$

Classical and Quantum P_{AB}

- Distribute $P_{AB}^{[ab]}$ to get high score

	$Q^{[0]}$		$Q^{[1]}$	
$P^{[0]}$	0.2	0	0.1	0.1
	0.8	0	0.4	0.4
$P^{[1]}$	0	0	0	0
	1	0	0.5	0.5

$$P_{AB} = PQ$$

Classical strategy

Φ	$V^{[0]}$		$V^{[1]}$	
$U^{[0]}$	0.43	0.07	0.07	0.43
	0.07	0.43	0.43	0.07
$U^{[1]}$	0.07	0.43	0.07	0.43
	0.43	0.07	0.43	0.07

$$P_{AB} = |\langle UV\Phi\rangle|^2$$

Quantum strategy

Classical Bayesian Nash

- Random play results in Negative Payoff
- Eight Nash E. : examples -->

$$\Pi^{\star}_{Ai} = \Pi^{\star}_{Bl} = 0$$

- Inequitable Split in BoS sector

$$\Pi^{[00]\star}_{Ai} = 0,\ \Pi^{[00]\star}_{Bl} = 0$$

$$\Pi^{[00]\star}_{Ai} = 3,\ \Pi^{[00]\star}_{Bl} = 1$$

$$\Pi^{[00]\star}_{Ai} = 1,\ \Pi^{[00]\star}_{Bl} = 3$$

1	0	1	0
0	0	0	0
0	0	0	0
1	0	1	0

0	1	0	1
0	0	0	0
0	1	0	1
0	0	0	0

$P^{[ab]}_{AB}$

Quantum Bayesian Nash

- Maximally entangled state

$$\gamma = \frac{\pi}{2}$$
$$\beta_0^\star - \alpha_0^\star = \pi/8 \qquad \tau = \frac{1}{2}\cos^2\frac{\pi}{8}$$
$$\beta_1^\star - \alpha_0^\star = -5\pi/8 \qquad \sigma = \frac{1}{2}\sin^2\frac{\pi}{8}$$
$$\beta_0^\star - \alpha_1^\star = 3\pi/8$$

τ	σ	σ	τ
σ	τ	τ	σ
σ	τ	σ	τ
τ	σ	τ	σ

$P_{AB}^{[ab]}$

- Beat classical logic

$$\Pi^\star_{Ai} = \Pi^\star_{Bl} = 4\frac{\sigma}{\sqrt{2}}$$

- Equitable Split in BoS sector

τ=0.427

σ=0.073

$$\Pi^{[00]\star}_{Ai} = \Pi^{[00]\star}_{Bl} = 4\tau$$

Bell Inequality

- Gedanken experiment on dichotomic 2 x 2 system

  Alice spin measured in settings $a = 0, 1$,
  projection $A = 0, 1$ $(s_A=(-1)^A)$

  Bob spin measured in settings $b = 0, 1$,
  projection $B = 0, 1$ $(s_B=(-1)^B)$

- With Local Realism, $P^{[ab]}_{AB}$ satisfy

$$P^{[00]}_{00} - P^{[10]}_{00} - P^{[01]}_{00} - P^{[11]}_{11} \leq 0$$

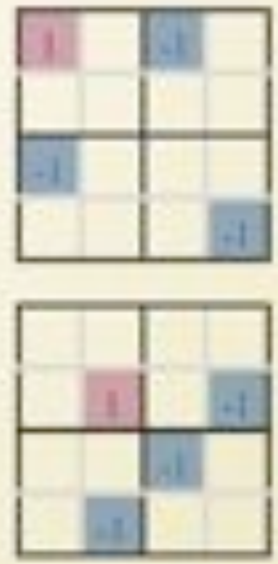
$$P^{[00]}_{11} - P^{[10]}_{11} - P^{[01]}_{11} - P^{[11]}_{00} \leq 0$$

Bell & Quantum Nash

- Payoff of Cereceda Game

$$\Pi_{Ai} = \frac{1}{4}(P_{00}^{[00]} - P_{00}^{[10]} - P_{00}^{[01]} - P_{11}^{[11]}) + \frac{3}{4}(P_{11}^{[00]} - P_{11}^{[10]} - P_{11}^{[01]} - P_{00}^{[11]})$$

$$\Pi_{Bl} = \frac{3}{4}(P_{00}^{[00]} - P_{00}^{[10]} - P_{00}^{[01]} - P_{11}^{[11]}) + \frac{1}{4}(P_{11}^{[00]} - P_{11}^{[10]} - P_{11}^{[01]} - P_{00}^{[11]})$$



- Positive payoffs are result of nonlocal strategy
- Never achieved with classical strategies

Anatomy of Quantum Move

- Identify $P_1^{[a]} = \sin^2 \alpha^{[a]},\ Q_1^{[b]} = \sin^2 \beta^{[b]}$

$$P_{00}^{[ab]} = \cos^2 \frac{\gamma}{2} P_0^{[a]} Q_0^{[b]} + \sin^2 \frac{\gamma}{2} P_1^{[a]} Q_1^{[b]} + \cos\phi \sin\gamma \sqrt{P_0^{[a]} P_1^{[a]} Q_0^{[b]} Q_1^{[b]}}$$

$$P_{10}^{[ab]} = \cos^2 \frac{\gamma}{2} P_1^{[a]} Q_0^{[b]} + \sin^2 \frac{\gamma}{2} P_0^{[a]} Q_1^{[b]} - \cos\phi \sin\gamma \sqrt{P_0^{[a]} P_1^{[a]} Q_0^{[b]} Q_1^{[b]}}$$

$$P_{01}^{[ab]} = \cos^2 \frac{\gamma}{2} P_0^{[a]} Q_1^{[b]} + \sin^2 \frac{\gamma}{2} P_1^{[a]} Q_0^{[b]} - \cos\phi \sin\gamma \sqrt{P_0^{[a]} P_1^{[a]} Q_0^{[b]} Q_1^{[b]}}$$

$$P_{11}^{[ab]} = \cos^2 \frac{\gamma}{2} P_1^{[a]} Q_1^{[b]} + \sin^2 \frac{\gamma}{2} P_0^{[a]} Q_0^{[b]} + \cos\phi \sin\gamma \sqrt{P_0^{[a]} P_1^{[a]} Q_0^{[b]} Q_1^{[b]}}$$

- 1st+2nd terms: Game-Symmetrizer / Altruism
- 3rd term: Quantum Interference / Nonlocality

$$|\Phi\rangle = \cos\frac{\gamma}{2}|00\rangle + e^{i\phi}\sin\frac{\gamma}{2}|11\rangle$$

Altruism and Nonlocality

- Altruism most visible in $\gamma = \pi/2, \ \phi = \pi/2$ case

$$P^{[ab]}_{AB} = \frac{1}{2} P^{[a]}_{A} Q^{[b]}_{B} + \frac{1}{2} P^{[a]}_{B} Q^{[b]}_{A} \qquad \text{(since } M^{[ab]}_{AB} = L^{[ab]}_{BA}\text{)}$$

$$\Pi^{[ab]}_{Ai} = \Pi^{[ab]}_{Bl} = \frac{1}{2} \sum_{A,B} (M^{[ab]}_{AB} + L^{[ab]}_{AB}) P^{[a]}_{A} Q^{[b]}_{B}$$

A local, thus classical correlation ("cheap talk")

- Nonlocal and altruistic in $\gamma = \pi/2, \ \phi = 0$ case

$$\Pi^{[ab]}_{Ai} = \sum_{A} M^{[ab]}_{AA} \cos^2(\alpha^{[a]} - \beta^{[b]}) + \sum_{A \neq B} M^{[ab]}_{AB} \sin^2(\alpha^{[a]} - \beta^{[b]})$$

Summary

- In joint probability formalism, Quantum Strategy is a natural extension of Classical Strategy
- Separation of control variable and probability
  -> Correlated and Nonlocal Strategies inclusive
- Concept of Control (strategy) and Gain (payoff) to Quantum Information and Quantum Metaphysics
- Quantum entanglement manifest itself in Harsanyi games with Bayesian players

FUTURE DIRECTIONS (GEN)

- Do quantum game experiment!
- N player quantum games
- Application in auction, finance?
- Quantum information processing! (proper 2-particle control to enhance desired phenomena)

References

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