

# Exotic Quantum Holonomy

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# Introduction

- ✱ Quantum (an)holonomy:
  - quantum resource for system control
  - **adiabatic** & **cyclic** variation of parameters
- ✱ **Berry** (nondegenerate) & **Wilczek-Zee** (degenerate)
- ✱ We point out existence of **their exotic sister**
- ✱ Key role of Berry-Mead **gauge connection**

# Conventional Theory

- Parametric system  $H(\alpha)$  with no degeneracy

$$H(\alpha) |\Psi_n(\alpha)\rangle = E_n(\alpha) |\Psi_n(\alpha)\rangle$$



$\alpha = [0, 2\pi) : 2\pi$ - periodic

- $C: \alpha = 0 \rightarrow 2\pi$  : adiabatic cyclic variation  $\alpha_t$

$$|\Psi_n\rangle \longrightarrow B(C) |\Psi_n\rangle e^{-i \int dt E_n(\alpha_t)} \quad \Psi_n = \Psi_n(0)$$

## Berry phase

$A_{n,n}(\alpha)$  : Abelian Berry – Mead connection

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$$B(C) = e^{i \oint_C d\alpha \langle \Psi_n(\alpha) | i \partial_\alpha | \Psi_n(\alpha) \rangle}$$

# Full Theory

- $C: \alpha = 0 \rightarrow 2\pi$  : adiabatic cyclic variation  $\alpha_t$

$$|\Psi_n\rangle \longrightarrow M(C) |\Psi_n\rangle e^{-i \int dt E_n(\alpha_t)}$$

$$= \sum_m |\Psi_m\rangle M_{m,n}(C) e^{-i \int dt E_n(\alpha_t)}$$

- $M(C) = W(C)B(C)$  - permutation with phase

$$B_{m,n}(C) = \left[ \overleftarrow{\mathcal{T}} e^{i \oint_C d\alpha A^D(\alpha)} \right]_{m,n}$$

$$W_{m,n}(C) = \left[ \overrightarrow{\mathcal{T}} e^{-i \oint_C d\alpha A(\alpha)} \right]_{m,n}$$

$$A_{m,n}(\alpha) = \langle \Psi_m(\alpha) | i\partial_\alpha | \Psi_n(\alpha) \rangle, \quad A_{m,n}^D(\alpha) = A_{m,n}(\alpha) \delta_{m,n}$$



# Gauge Invariance

- Gauge transformation  $|\Psi_n(\alpha)\rangle \longrightarrow e^{ig_n(\alpha)} |\Psi_n(\alpha)\rangle$

$$A_{m,n} \longrightarrow A_{m,n} e^{i(g_m(\alpha) - g_n(\alpha))} - \delta_{m,n} g_n(\alpha)$$

$$W_{m,n} \longrightarrow W_{m,n} e^{-i\delta_{m,n} g_n(\alpha)} \quad B_{m,n} \longrightarrow B_{m,n} e^{i\delta_{m,n} g_n(\alpha)}$$

$$|\Psi_m\rangle M_{m,n} \langle\Psi_n| \longrightarrow e^{ig_m} |\Psi_m\rangle M_{m,n} \langle\Psi_n| e^{-ig_n} \text{ invariant}$$

- For Berry phase, we can choose  $W_{m,n} = \delta_{m,n}$

$$M_{m,n}(C) = \overleftarrow{\mathcal{T}} e^{i \oint_C d\alpha A_{m,m}^D(\alpha)} \cdot \delta_{m,n}$$

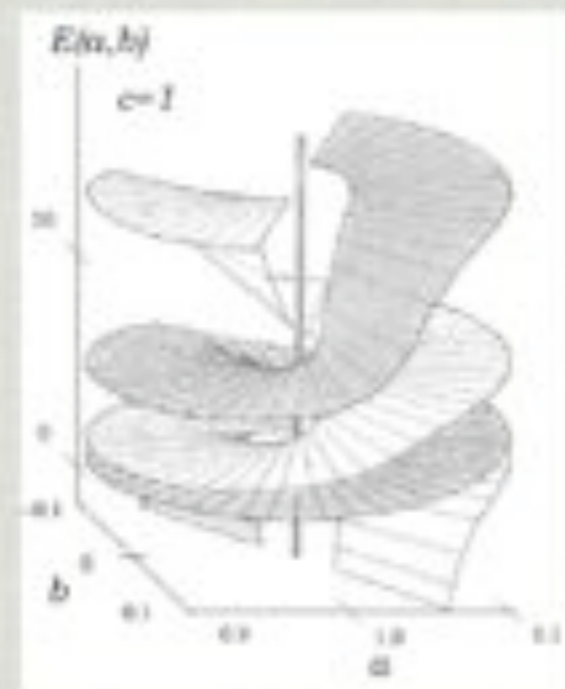
- With **parallel transport** gauge  $A_{m,n}^D = 0$

$$M_{m,n}(C) = \left[ \overrightarrow{\mathcal{T}} e^{-i \oint_C d\alpha A(\alpha)} \right]_{m,n} \quad : \text{new expression!}$$

# Nondiagonal Holonomy

- Holonomy  $M$  **not** necessarily diagonal
  - For two state system, for example,
    - \*  $M_{m,n} = m_0 I + m_3 \sigma_3$  : Berry phase
    - \*  $M_{m,n} = m_1 \sigma_1 + m_2 \sigma_2$  : *To another eigenstate!*

- Spiral holonomy
  - 1D system with **point interaction**  
(Cheon 1998)
  - periodically **kicked spin 1/2**  
(Tanaka & Miyamoto 2007)



# Outline of Proof

- Solve the parametric time evolution

$$\left( H(\alpha_t) - i \frac{d}{dt} \right) |\Psi(t)\rangle = 0$$

- Formal solution  $|\Psi(t)\rangle = U(t) |\Psi(0)\rangle$

$$U(t) = \sum_{n,m} |v_n(\alpha_t)\rangle \left[ \overleftarrow{\mathcal{T}} e^{-i \int_0^t dt F(\alpha_t)} \right]_{n,m} \langle v_m(\alpha_0)|$$

- Fujikawa matrix  $\{|v_n(\alpha_t)\rangle\}$  : arbitrary basis

$$F_{m,n}(\alpha) = \langle v_m(\alpha) | \{ H(\alpha) - \dot{\alpha} i \partial_{\alpha} \} | v_n(\alpha) \rangle$$

# Outline of Proof (cont'd)

- With identity

$$|v_m(\alpha_t)\rangle = \sum_l |v_l(\alpha_0)\rangle \left[ \overrightarrow{\mathcal{T}} e^{-i \int_{\alpha_0}^{\alpha_t} d\alpha A(\alpha)} \right]_{l,m}$$

- we have

$$U(t) = \sum_{n,m} |v_n(\alpha_0)\rangle [W(\alpha_t, \alpha_0) D(\alpha_t, \alpha_0)]_{n,m} \langle v_m(\alpha_0)|$$

with

$$W_{m,n}(\alpha_t, \alpha_0) = \left[ \overrightarrow{\mathcal{T}} e^{-i \int_{\alpha_0}^{\alpha_t} d\alpha A(\alpha)} \right]_{m,n}$$

$$D_{m,n}(\alpha_t, \alpha_0) = \left[ \overleftarrow{\mathcal{T}} e^{-i \int_0^t dt F(\alpha_t)} \right]_{m,n}$$

# Outline of Proof (complt)

- If diagonal elements dominate in  $F_{m,n}$

$$\begin{aligned} F_{m,n}(\alpha) &= \langle v_m(\alpha) | \{H(\alpha) - \dot{\alpha} i \partial_\alpha\} | v_n(\alpha) \rangle \\ &\approx \delta_{m,n} \langle v_n(\alpha) | \{H(\alpha) - \dot{\alpha} i \partial_\alpha\} | v_n(\alpha) \rangle \end{aligned}$$

- with  $v_n(\alpha) = \Psi_n(\alpha)$ , and assuming adiabaticity,

$$\begin{aligned} D_{m,n}(\alpha_t, \alpha_0) &\approx \delta_{m,n} \mathcal{T} \left[ e^{-i \int_0^t dt \langle v_n(\alpha) | H(\alpha) | v_n(\alpha) \rangle} e^{i \int_{\alpha_0}^{\alpha_t} d\alpha \langle v_n(\alpha) | i \partial_\alpha v_n(\alpha) \rangle} \right] \\ &= \delta_{m,n} \mathcal{T} e^{i \int_{\alpha_0}^{\alpha_t} d\alpha A_{n,n}(\alpha)} e^{-i \int_0^t dt E_n(\alpha)} = B_{m,n} e^{-i \int_0^t dt E_n(\alpha)} \end{aligned}$$

# Comments

- Valid whenever  $F_{m,n}$  is approximated (or is) diagonal
  - $v_n(\alpha)$ : instantaneous basis with adiabatic variation  $\rightarrow$  Berry
  - $\alpha=t$ ,  $v_n(t)$  exact solution of the problem  $\rightarrow$  Aharonov-Anandan
- After diagonal approx,  
 $U(N)$  gauge symmetry  $\rightarrow$  “diagonal” invariance

# Degenerate Case

- $C: \alpha = 0 \rightarrow 2\pi$  : adiabatic cyclic variation  $\alpha_t$

$$\begin{aligned} |\Psi_{nj}\rangle &\longrightarrow M(C) |\Psi_{nj}\rangle e^{-i \int dt E_n(\alpha_t)} \\ &= \sum_{mi} |\Psi_{mi}\rangle M_{mi,nj}(C) e^{-i \int dt E_n(\alpha_t)} \end{aligned}$$

- $M(C) = W(C)B(C)$  : block permutation

$$B_{mi,nj}(C) = \left[ \overleftarrow{\mathcal{T}} e^{i \oint_C d\alpha A^D(\alpha)} \right]_{mi,nj}$$

$$W_{mi,nj}(C) = \left[ \overrightarrow{\mathcal{T}} e^{-i \oint_C d\alpha A(\alpha)} \right]_{mi,nj}$$

$$A_{mi,nj}^D(\alpha) = A_{ni,nj}(\alpha) \delta_{m,n}$$

# Kicked Spin 1/2

- Time-periodic two-level system

$$H = T\sigma_3 + \lambda V \sum_{n=-\infty}^{\infty} \delta(t - n - \frac{1}{2})$$

with

$$V = \frac{p}{2} + \frac{2-p}{2} \sum_{i=1}^3 b_i \sigma_i \quad \left. \begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix} \right\} = \begin{cases} \sin \gamma \cos \xi \\ \sin \gamma \sin \xi \\ \cos \gamma \end{cases}$$

- parameter space

$$\{\lambda, \gamma, \xi\} = S^1 \times S^2$$

- one-step evolution

$$U_V = e^{-\frac{iT}{2}\sigma_3} e^{-i\lambda V} e^{-\frac{iT}{2}\sigma_3}$$

- quasi-eigen eq.

$$U_V |\Psi_n\rangle = e^{-i\varepsilon_n T} |\Psi_n\rangle$$

# Full Solution ( $J=1/2$ )

- Quasenergy  $\varepsilon_n = \frac{p}{2}\lambda + (-)^n E_{\frac{2-p}{2}\lambda, \gamma, T}$

- Quasieigenstate

$$|\Psi_0\rangle = \begin{pmatrix} e^{-\frac{i\xi}{2}} \cos q \\ e^{\frac{i\xi}{2}} \sin q \end{pmatrix} \quad |\Psi_1\rangle = \begin{pmatrix} -e^{-\frac{i\xi}{2}} \sin q \\ e^{\frac{i\xi}{2}} \cos q \end{pmatrix}$$

$$q = Q_{\frac{2-p}{2}\lambda, \gamma, T}$$

- with  $E_{\lambda, \gamma, T} = \arccos(\cos \lambda \cos T - \sin \lambda \sin T \cos \gamma)$

$$Q_{\lambda, \gamma, T} = \frac{1}{2} \arctan \left( \frac{\sin \lambda \sin \gamma}{\cos \lambda \sin T - \sin \lambda \cos T \cos \gamma} \right)$$

# Gauge Field ( $J=1/2$ )

- With the representation  $|\Phi_0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$   $|\Phi_1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$   
gauge connections are

$$A^\xi = \frac{1}{2}(\Sigma_3 \cos \gamma - \Sigma_1 \sin \gamma)$$

$$A^\gamma = \partial_\gamma Q_{\frac{2-p}{2}, \lambda, \gamma, T} \Sigma_2$$

$$A^\lambda = \partial_\lambda Q_{\frac{2-p}{2}, \lambda, \gamma, T} \Sigma_2$$

- diagonal path-ordered integral of  $A$

$$B(C^\xi) = e^{i\pi \cos \gamma \Sigma_3} \quad B(C^\gamma) = 1 \quad B(C^\lambda) = 1$$

# Holonomy ( $J=1/2$ )

- Path-ordered integral of full A

$$W(C^\xi) = \cos \pi - i(\Sigma_3 \cos \gamma - \Sigma_1 \sin \gamma) \sin \pi$$

$$W(C^\gamma) = \cos Q_{\frac{(2-p)\lambda}{2}, 2\pi, T} - i\Sigma_2 \sin Q_{\frac{(2-p)\lambda}{2}, 2\pi, T}$$

$$W(C^\lambda) = \cos Q_{(2-p)\pi, \gamma, T} - i\Sigma_2 \sin Q_{(2-p)\pi, \gamma, T}$$

- Holonomy

**MONOPOLE (FOR  $T=0$ )**

$$M(C_\xi) = e^{-i\pi(1-\Sigma_3 \cos \gamma)} \quad M(C_\gamma) = -1$$

$$M(C_\lambda) = \cos \frac{(2-p)\pi}{2} - i\Sigma_2 \sin \frac{(2-p)\pi}{2}$$

# Spiral Holonomy ( $J=1/2$ )

- Periodicity of  $Q$

$$Q_{\lambda+\pi, \gamma, T} = Q_{\lambda, \gamma, T} + \frac{\pi}{2}$$

and of  $\varepsilon$

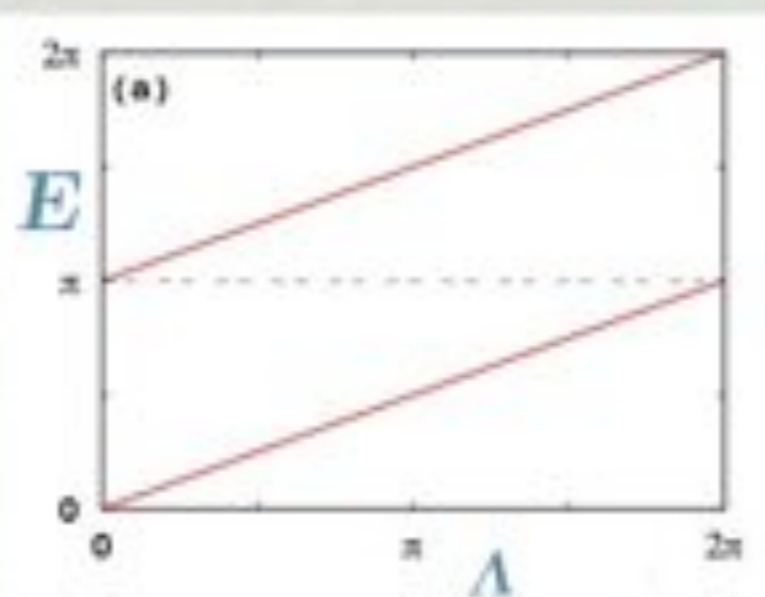
$$\varepsilon_n(\lambda + 2\pi) = \varepsilon_{n+p}(\lambda)$$

- Level shift with cyclic parameter variation

$$\Delta n(C_\xi) = \Delta n(C_\gamma) = 0$$

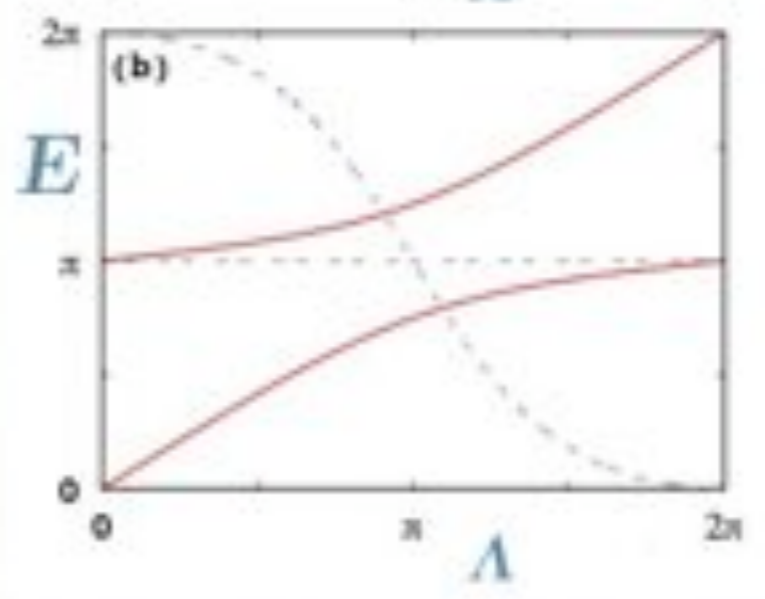
$$\Delta n(C_\lambda) = p$$

states alternate with odd  $p$



$P=1$

$\gamma = \pi/4$



$P=1$

$\gamma = \pi/8$

# Kicked Spin 3/2

- Pairwise doubly degenerate 4 levels

$$H = T\tau_5 + \lambda V \sum_{n=-\infty}^{\infty} \delta(t - n - \frac{1}{2})$$

with

$$V = \frac{p}{2} + \frac{2-p}{2} \sum_{i=1}^5 b_i \tau_i$$

$$\left. \begin{matrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{matrix} \right\} = \begin{cases} \sin \gamma \cos \eta \cos \xi \\ \sin \gamma \cos \eta \sin \xi \\ \sin \gamma \sin \eta \cos \zeta \\ \sin \gamma \sin \eta \sin \zeta \\ \cos \gamma \end{cases}$$

- 3/2 spin matrices

$$\tau_1 = \begin{pmatrix} \mathbf{0} & i\sigma_2 \\ -i\sigma_2 & \mathbf{0} \end{pmatrix} \quad \tau_2 = \begin{pmatrix} \mathbf{0} & -i\sigma_1 \\ i\sigma_1 & \mathbf{0} \end{pmatrix}$$

$$\tau_3 = \begin{pmatrix} \mathbf{0} & I \\ I & \mathbf{0} \end{pmatrix} \quad \tau_4 = \begin{pmatrix} \mathbf{0} & -i\sigma_3 \\ i\sigma_3 & \mathbf{0} \end{pmatrix} \quad \tau_5 = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & -I \end{pmatrix}$$

- parameter space

$$\{\lambda, \gamma, \eta, \xi, \zeta\} = S^1 \times S^4$$

# Full Solution ( $J=3/2$ )

- Quasienergy  $\varepsilon_{nj} = \frac{p}{2}\lambda + (-)^n E_{\frac{2-p}{2}\lambda, \gamma, T}$

- Quasieigenstates

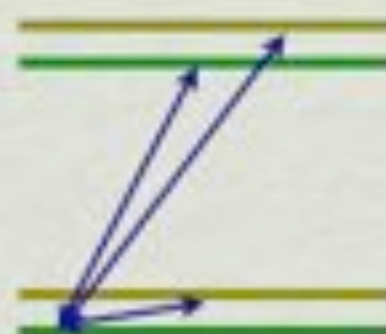
$$\theta_{\pm} = \frac{\xi \pm \zeta}{2}$$

$$|\Psi_{00}\rangle = \begin{pmatrix} e^{-i\theta + \cos q} \\ 0 \\ e^{-i\theta - \sin \eta \sin q} \\ e^{i\theta - \cos \eta \sin q} \end{pmatrix} \quad |\Psi_{01}\rangle = \begin{pmatrix} 0 \\ e^{i\theta + \cos q} \\ -e^{-i\theta - \cos \eta \sin q} \\ e^{i\theta - \sin \eta \sin q} \end{pmatrix} \quad q = Q_{\frac{2-p}{2}\lambda, \gamma, T}$$

$$|\Psi_{10}\rangle = \begin{pmatrix} -e^{-i\theta + \sin \eta \sin q} \\ e^{i\theta + \cos \eta \sin q} \\ e^{-i\theta - \cos q} \\ 0 \end{pmatrix} \quad |\Psi_{11}\rangle = \begin{pmatrix} -e^{-i\theta + \cos \eta \sin q} \\ -e^{i\theta + \sin \eta \sin q} \\ 0 \\ e^{i\theta - \cos q} \end{pmatrix}$$

# Holonomy All Star

- “Yang-Mills instanton” for  $S^4$  sector
- Mixture of Wilczek-Zee and spiral holonomies for  $S^1$  sector



$$A^\lambda = \partial_\lambda Q_{\frac{2-p}{2}\lambda, \gamma, T} \begin{bmatrix} \mathbf{0} & -i \sin \eta I + \cos \eta \Sigma_2 \\ i \sin \eta I + \cos \eta \Sigma_2 & \mathbf{0} \end{bmatrix}$$

$$M(C_\lambda) = \cos \frac{(2-p)\pi}{2} - i \sin \frac{(2-p)\pi}{2} \left( \sin \eta \begin{bmatrix} 0 & -iI \\ iI & 0 \end{bmatrix} + \cos \eta \begin{bmatrix} 0 & \Sigma_2 \\ \Sigma_2 & 0 \end{bmatrix} \right)$$

# Summary

- Fully gauge invariant formulation of quantum holonomy
- All known quantum holonomies under its fold
- Potentially useful spiral holonomy legitimized
- Illustrative examples including 4 level-system displaying all holonomies mixed together

# References

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# Appendix I: $H(t+T)=H(t)$

- “Time-domain Bloch” theorem

$$i \partial_t \Psi(t) = H(t) \Psi(t) \rightarrow i \partial_t \Psi(t+T) = H(t) \Psi(t+T)$$

*quasi-eigenstates*  $\Psi(t+T) \propto \Psi(t)$  exist

- Period evolution  $U(T) = \mathcal{T} \exp\{ -i \int^T dt H(t) \}$

$$U(T) \Psi_n(t) = \exp\{ i E_n T \} \Psi_n(t)$$

- *quasi-eigenvalue*  $E_n$  ;  $[0, 2\pi/T]$  periodic
- Hamiltonian system obtained as  $T \rightarrow 0$

# Appendix II: $S^1$ -space

- Evolution operator  $U_V(\lambda) = e^{-\frac{iT}{2}\sigma_3} e^{-i\lambda V} e^{-\frac{iT}{2}\sigma_3}$  is  $\lambda$ -periodic  $U_V(\lambda+2\pi) = U_V(\lambda)$ , if

$$e^{-2\pi i V} = 1$$

- $V = b_0 + b_s \sigma$  ( $\sigma \cdot \sigma = 1$ )

$$e^{-2\pi i b_0} (\cos 2\pi b_s + i\sigma \sin 2\pi b_s) = 1$$

$$\longrightarrow b_0 = n/2, \quad b_s = 1 - n/2$$

# Appendix III: Diabolical?

- Johanssen-Sjöqvist theorem

if holonomy on  $dS \rightarrow E(\alpha)$  : degeneracy on  $S$   
 $\rightarrow A(\alpha)$  : singularity on  $S$

- Diabolical point for Berry phase  
on 2-parameter space  $S^2$  etc.



- What about Spiral Holonomy

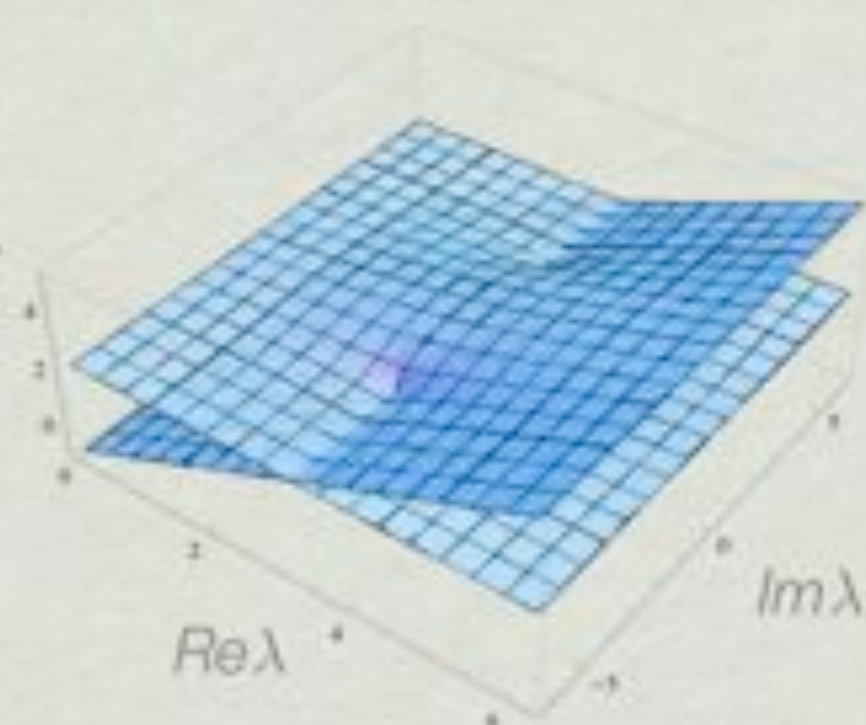
- found for parameter space  $S^1$  only
- hint: complex parameter space singularity?  
--- Exceptional Point

# Exceptional Point

- Ex. ; J=1/2 kicked spin,  $T=\pi/2$

- Square root singularity for  $Re\epsilon$

$$\epsilon_n(\lambda) = \frac{\lambda}{2} + (-1)^n \arccos \left( -\sin \frac{\lambda}{2} \cos \gamma \right)$$



- Simple pole for  $A_\lambda(\lambda)$

$$A_\lambda(\lambda) = \frac{\sin \gamma}{4(\cos \frac{\lambda}{2} - i \sin \frac{\lambda}{2} \sin \gamma)(\cos \frac{\lambda}{2} + i \sin \frac{\lambda}{2} \sin \gamma)} \sigma_2$$

$$\text{pole } \lambda_{\pm}^* = \pm 2 \arctan \frac{1}{i \sin \gamma} \quad \text{residue } \mu_{\pm} = \pm \frac{1}{4i}$$

$$M(C_\lambda) = \mathcal{T} e^{-i \int_0^{2\pi} d\lambda A_\lambda} = e^{i \frac{\pi}{2} \sigma_2} = -i \sigma_2$$

