New Anatomy of Quantum Holonomy

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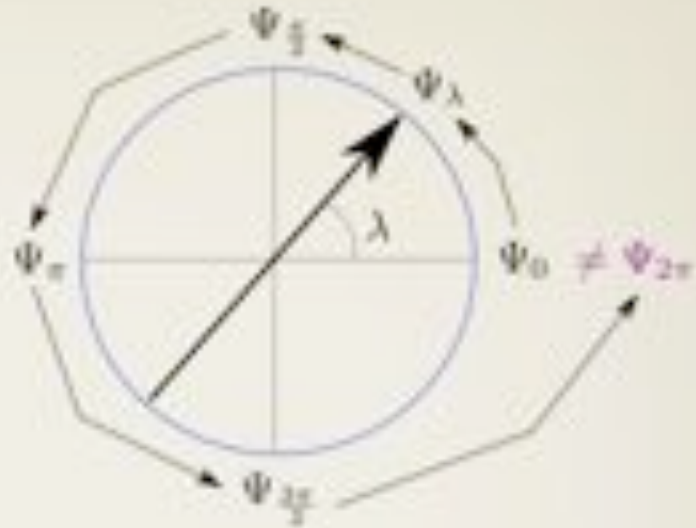
Introduction



* Quantum (an)holonomy:
  - quantum resource for system control
  - adiabatic & cyclic variation of parameters
* In adittion to Berry & Wilczek-Zee holonomy, we point out existence of their exotic sister
* Key role of Berry-Mead gauge connection
* Unifyied formulation of quantum holonomies

Conventional Theory

* Parametric system $H(\alpha)$ with no degeneracy

$$H(\alpha)\,|\Psi_n(\alpha)\rangle = E_n(\alpha)\,|\Psi_n(\alpha)\rangle$$

α =[0, 2π) : 2π- periodic

* C: α =0→2π : adiabatic cyclic variation α_t

$$|\Psi_n\rangle \longrightarrow B(C)\,|\Psi_n\rangle\, e^{-i\int dt E_n(\alpha_t)} \qquad \Psi_n = \Psi_n(0)$$

Berry phase

$A_{n,n}(\alpha)$: Abelian Berry – Mead connection

$$B(C) = e^{i\oint_C d\alpha \langle \Psi_n(\alpha)|i\partial_\alpha|\Psi_n(\alpha)\rangle}$$

Full Theory

- C: α =0→2π : adiabatic cyclic variation α_t

  $$|\Psi_n\rangle \longrightarrow M(C)\,|\Psi_n\rangle\, e^{-i\int dt E_n(\alpha_t)}$$
  $$= \sum_m |\Psi_m\rangle\, M_{m,n}(C) e^{-i\int dt E_n(\alpha_t)}$$

- $M(C) = W(C)B(C)$ - permutation with phase

$$B_{m,n}(C) = \left[\underset{\leftarrow}{\mathcal{T}} e^{i\oint_C d\alpha A^D(\alpha)}\right]_{m,n}$$

$$W_{m,n}(C) = \left[\underset{\rightarrow}{\mathcal{T}} e^{-i\oint_C d\alpha A(\alpha)}\right]_{m,n}$$

$$A_{m,n}(\alpha) = \langle\Psi_m(\alpha)|\, i\partial_\alpha\, |\Psi_n(\alpha)\rangle,\; A^D_{m,n}(\alpha) = A_{m,n}(\alpha)\delta_{m,n}$$

Gauge Invariance

- Gauge transformation $|\Psi_n(\alpha)\rangle \longrightarrow e^{ig_n(\alpha)}|\Psi_n(\alpha)\rangle$

$$A_{m,n} \longrightarrow A_{m,n}e^{i(g_m(\alpha)-g_n(\alpha))} - \delta_{m,n}g_n(\alpha)$$

$$W_{m,n} \longrightarrow W_{m,n}e^{-i\delta_{m,n}g_n(\alpha)} \qquad B_{m,n} \longrightarrow B_{m,n}e^{i\delta_{m,n}g_n(\alpha)}$$

$$|\Psi_m\rangle M_{m,n}\langle\Psi_n| \longrightarrow e^{ig_m}|\Psi_m\rangle M_{m,n}\langle\Psi_n|e^{-ig_n}$$ invariant

- For Berry phase, we can choose $W_{mn} = \delta_{mn}$

$$M_{m,n}(C) = \underleftarrow{\mathcal{T}}e^{i\oint_C d\alpha A^D_{m,m}(\alpha)} \cdot \delta_{m,n}$$

- With parallel transport gauge $A^D{}_{mn} = 0$

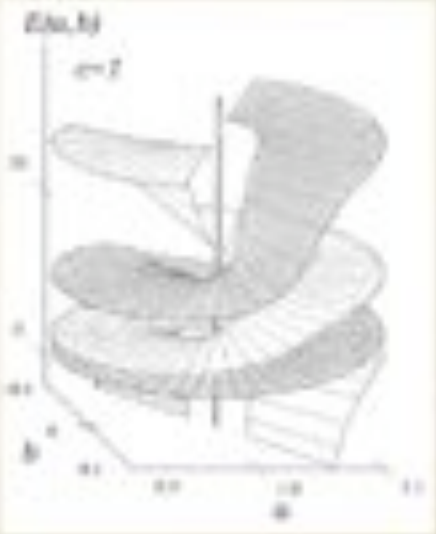
$$M_{m,n}(C) = \left[\underrightarrow{\mathcal{T}}e^{-i\oint_C d\alpha A(\alpha)}\right]_{m,n}$$: new expression!

Nondiagonal Holonomy

- Holonomy *M* **not** necessarily diagonal
  - For two state system, for example,
    - $M_{m,n} = m_0 I + m_3\sigma_3$: Berry phase
    - $M_{m,n} = m_1\sigma_1 + m_2\sigma_2$: *to another eigenstate*!
- Spiral holonomy
  - 1D system with point interaction

    (Cheon 1998)
  - periodically kicked spin 1/2

    (Tanaka & Miyamoto 2007)



Outline of Proof

* Solve the parametric time evolution

$$\left(H(\alpha_t) - i\frac{d}{dt}\right)|\Psi(t)\rangle = 0$$

$$|\Psi(t)\rangle = \sum_m |v_m(\alpha_t)\rangle\, b_m(t)$$

$$i\dot{b}_m(t) = \sum_n F_{m,n}(\alpha_t) b_n(t)$$

* Formal solution $|\Psi(t)\rangle = U(t)\,|\Psi(0)\rangle$

$$U(t) = \sum_{n,m} |v_n(\alpha_t)\rangle \left[\underset{\leftarrow}{\mathcal{T}} e^{-i\int_0^t dt F(\alpha_t)}\right]_{n,m} \langle v_m(\alpha_0)|$$

$\{|v_n(\alpha_t)\rangle\}$: arbitrary basis

* Fujikawa matrix

$$F_{m,n}(\alpha) = \langle v_m(\alpha)|\left\{H(\alpha) - \dot{\alpha}\, i\partial_\alpha\right\}|v_n(\alpha)\rangle$$

Outline of Proof (cont'd)

* With identity

$$|v_m(\alpha_t)\rangle = \sum_l |v_l(\alpha_0)\rangle \left[\underrightarrow{\mathcal{T}} e^{-i\int_{\alpha_0}^{\alpha_t} d\alpha A(\alpha)}\right]_{l,m}$$

* we have

$$A_{m,n}(\alpha) = \langle v_m(\alpha)| i\partial_\alpha |v_n(\alpha)\rangle$$

$$U(t) = \sum_{n,m} |v_n(\alpha_0)\rangle \left[W(\alpha_t,\alpha_0)D(\alpha_t,\alpha_0)\right]_{n,m} \langle v_m(\alpha_0)|$$

with

$$W_{m,n}(\alpha_t,\alpha_0) = \left[\underrightarrow{\mathcal{T}} e^{-i\int_{\alpha_0}^{\alpha_t} d\alpha A(\alpha)}\right]_{m,n}$$

$$D_{m,n}(\alpha_t,\alpha_0) = \left[\underleftarrow{\mathcal{T}} e^{-i\int_0^t dt F(\alpha_t)}\right]_{m,n}$$

Outline of Proof (complt)

* If diagonal elements dominate in $F_{m,n}$

$$F_{m,n}(\alpha) = \langle v_m(\alpha)| \{H(\alpha) - \dot{\alpha}\, i\partial_\alpha\} |v_n(\alpha)\rangle$$
$$\approx \delta_{m,n} \langle v_n(\alpha)| \{H(\alpha) - \dot{\alpha}\, i\partial_\alpha\} |v_n(\alpha)\rangle$$

* with $v_n(\alpha) = \Psi_n(\alpha)$, and assuming adiabaticity,

$$D_{m,n}(\alpha_t, \alpha_0)$$
$$\approx \delta_{m,n} \underleftarrow{\mathcal{T}} \left[e^{-i\int_0^t dt \langle v_n(\alpha)|H(\alpha)|v_n(\alpha)\rangle} e^{i\int_{\alpha_0}^{\alpha_t} d\alpha \langle v_n(\alpha)|i\partial_\alpha v_n(\alpha)\rangle} \right]$$
$$= \delta_{m,n} \underleftarrow{\mathcal{T}} e^{i\int_{\alpha_0}^{\alpha_t} d\alpha A_{n,n}(\alpha)} e^{-i\int_0^t dt E_n(\alpha)} = B_{m,n} e^{-i\int_0^t dt E_n(\alpha)}$$

Comments

* Valid whenever $F_{m,n}$ is approximated (or is) diagonal
  - $v_n(\alpha)$: instantaneous basis with adiabatic variation $\rightarrow$ Berry
  - $\alpha = t$, $v_n(t)$ exact solution of the problem $\rightarrow$ Aharonov-Anandan
* After diagonal approx,

  $U(N)$ gauge symmetry $\rightarrow$ "diagonal" invariance

Degenerate Case

- C: α =0→2π : adiabatic cyclic variation α_t

$$|\Psi_{nj}\rangle \longrightarrow M(C)\,|\Psi_{nj}\rangle\, e^{-i\int dt E_n(\alpha_t)}$$
$$= \sum_{mi} |\Psi_{mi}\rangle\, M_{mi,nj}(C) e^{-i\int dt E_n(\alpha_t)}$$

- $M(C) = W(C)B(C)$: block permutation

$$B_{mi,nj}(C) = \left[\underset{\leftarrow}{\mathcal{T}} e^{i\oint_C d\alpha A^D(\alpha)}\right]_{mi,nj}$$

$$W_{mi,nj}(C) = \left[\underset{\rightarrow}{\mathcal{T}} e^{-i\oint_C d\alpha A(\alpha)}\right]_{mi,nj}$$

$$A^D_{mi,nj}(\alpha) = A_{ni,nj}(\alpha)\delta_{m,n}$$

T-Periodic System $H(t+T)=H(t)$

* "Time-domain Bloch" theorem

  $i\,\partial_t\,\Psi(t) = H(t)\,\Psi(t) \rightarrow i\,\partial_t\,\Psi(t+T) = H(t)\,\Psi(t+T)$

  quasi-eigenstates $\Psi(t+T) \propto \Psi(t)$ exist

* Period evolution $U(T) = \mathcal{T}\exp\{ -i \int^T dt\, H(t) \}$

  $U(T)\,\Psi_n(t) = \exp\{ i\, E_n T \}\,\Psi_n(t)$

* *quasi-eigenvalue* E_n ; $[0, 2\pi/T]$ periodic

* Hamiltonian system obtained as $T \rightarrow 0$

Kicked Spin 1/2

* Time-periodic two-level system

$$H = T\sigma_3 + \lambda V \sum_{n=-\infty}^{\infty} \delta(t - n - \frac{1}{2})$$

with

$$V = \frac{p}{2} + \frac{2-p}{2} \sum_{i=1}^{3} b_i \sigma_i \qquad \left.\begin{matrix} b_1 \\ b_2 \\ b_3 \end{matrix}\right\} = \left\{ \begin{matrix} \sin\gamma\cos\xi \\ \sin\gamma\sin\xi \\ \cos\gamma \end{matrix}\right.$$

* parameter space

$$\{\lambda, \gamma, \xi\} = S^1 \times S^2$$

* one-step evolution

$$U_V = e^{-\frac{iT}{2}\sigma_3} e^{-i\lambda V} e^{-\frac{iT}{2}\sigma_3}$$

* quasi-eigen eq.

$$U_V \left|\Psi_n\right\rangle = e^{-i\varepsilon_n T} \left|\Psi_n\right\rangle$$

S^1 Parameter Space

* Evolution operator $U_V(\lambda) = e^{-\frac{iT}{2}\sigma_3} e^{-i\lambda V} e^{-\frac{iT}{2}\sigma_3}$

  is λ-peridic $U_V(\lambda+2\pi) = U_V(\lambda)$, if

$$e^{-2\pi i V} = 1$$

* $V = b_0 + b_s \sigma \qquad (\sigma \cdot \sigma = 1)$

$$e^{-2\pi i b_0}(\cos 2\pi b_s + i\sigma \sin 2\pi b_s) = 1$$

$$\longrightarrow \quad b_0 = n/2, \quad b_s = 1 - n/2$$

Full Solution (J=1/2)

- Quasinergy $\varepsilon_n = \frac{p}{2}\lambda + (-)^n E_{\frac{2-p}{2}\lambda,\gamma,T}$
- Quasieigenstate

$$|\Psi_0\rangle = \begin{pmatrix} e^{-\frac{i\xi}{2}}\cos q \\ e^{\frac{i\xi}{2}}\sin q \end{pmatrix} \quad |\Psi_1\rangle = \begin{pmatrix} -e^{-\frac{i\xi}{2}}\sin q \\ e^{\frac{i\xi}{2}}\cos q \end{pmatrix}$$

$$q = Q_{\frac{2-p}{2}\lambda,\gamma,T}$$

- with $E_{\lambda,\gamma,T} = \arccos\left(\cos\lambda\cos T - \sin\lambda\sin T\cos\gamma\right)$

$$Q_{\lambda,\gamma,T} = \frac{1}{2}\arctan\left(\frac{\sin\lambda\sin\gamma}{\cos\lambda\sin T - \sin\lambda\cos T\cos\gamma}\right)$$

Gauge Field (J=1/2)

- With the representation $|\Phi_0\rangle = \begin{bmatrix} 1 \\ 0 \end{bmatrix} |\Phi_1\rangle = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ gauge connections are

$$A^\xi = \frac{1}{2}(\Sigma_3 \cos\gamma - \Sigma_1 \sin\gamma)$$

$$A^\gamma = \partial_\gamma Q_{\frac{2-p}{2}\lambda,\gamma,T}\Sigma_2$$

$$A^\lambda = \partial_\lambda Q_{\frac{2-p}{2}\lambda,\gamma,T}\Sigma_2$$

- diagonal path-ordered integral of A

$$B(C^\xi) = e^{i\pi\cos\gamma\Sigma_3} \quad B(C^\gamma) = 1 \quad B(C^\lambda) = 1$$

Holonomy (J=1/2)

* Path-ordered integral of full A

$$W(C^{\xi}) = \cos\pi - i(\Sigma_3\cos\gamma - \Sigma_1\sin\gamma)\sin\pi$$

$$W(C^{\gamma}) = \cos Q_{\frac{(2-p)\lambda}{2},2\pi,T} - i\Sigma_2 \sin Q_{\frac{(2-p)\lambda}{2},2\pi,T}$$

$$W(C^{\lambda}) = \cos Q_{(2-p)\pi,\gamma,T} - i\Sigma_2 \sin Q_{(2-p)\pi,\gamma,T}$$

* Holonomy

monopole (for T=0)

$$M(C_{\xi}) = e^{-i\pi(1-\Sigma_3\cos\gamma)} \qquad M(C_{\gamma}) = -1$$

$$M(C_{\lambda}) = \cos\frac{(2-p)\pi}{2} - i\Sigma_2\sin\frac{(2-p)\pi}{2}$$

Spiral Holonomy (J=1/2)

* Periodicity of Q

$$Q_{\lambda+\pi,\gamma,T} = Q_{\lambda,\gamma,T} + \frac{\pi}{2}$$

and of ε

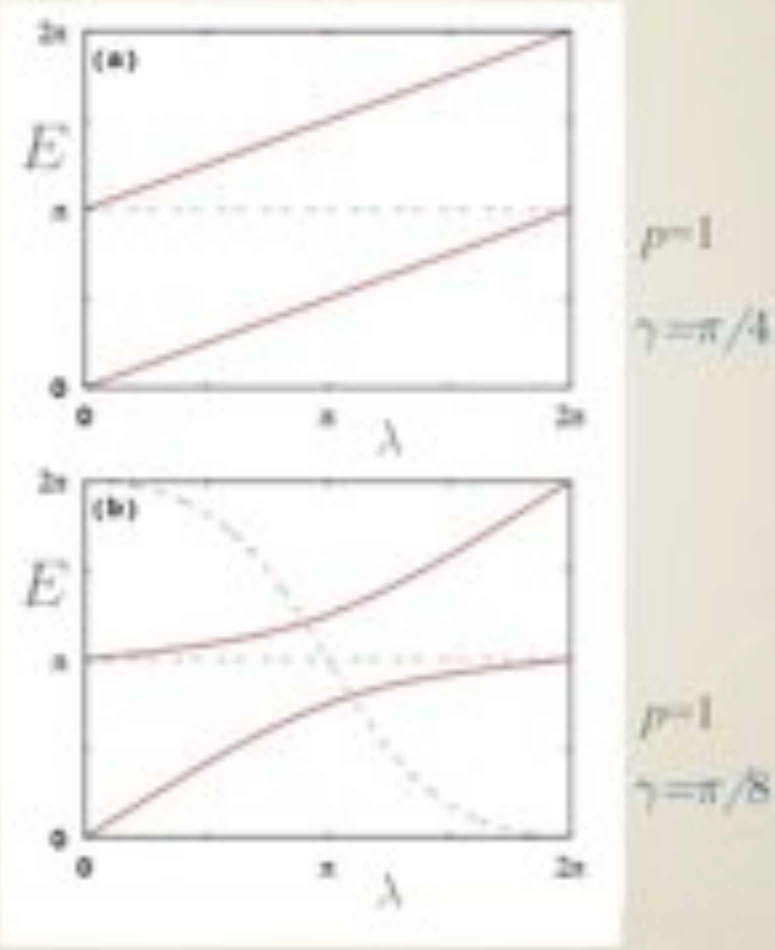
$$\varepsilon_n(\lambda + 2\pi) = \varepsilon_{n+p}(\lambda)$$

* Level shift with cyclic parameter variation

$$\Delta n(C_\xi) = \Delta n(C_\gamma) = 0$$

$$\Delta n(C_\lambda) = p$$

states alternate with odd p



Kicked Spin 3/2

* Pairwise doubly degenerate 4 levels

$$H = T\tau_5 + \lambda V \sum_{n=-\infty}^{\infty} \delta(t - n - \frac{1}{2})$$

with

$$V = \frac{p}{2} + \frac{2-p}{2}\sum_{i=1}^{5} b_i \tau_i$$

$$\left.\begin{matrix} b_1 \\ b_2 \\ b_3 \\ b_4 \\ b_5 \end{matrix}\right\} = \left\{\begin{matrix} \sin\gamma\cos\eta\cos\xi \\ \sin\gamma\cos\eta\sin\xi \\ \sin\gamma\sin\eta\cos\zeta \\ \sin\gamma\sin\eta\sin\zeta \\ \cos\gamma \end{matrix}\right.$$

* 3/2 spin matrices

$$\tau_1 = \begin{pmatrix} \mathbf{0} & i\sigma_2 \\ -i\sigma_2 & \mathbf{0} \end{pmatrix} \quad \tau_2 = \begin{pmatrix} \mathbf{0} & -i\sigma_1 \\ i\sigma_1 & \mathbf{0} \end{pmatrix}$$

$$\tau_3 = \begin{pmatrix} \mathbf{0} & I \\ I & \mathbf{0} \end{pmatrix} \quad \tau_4 = \begin{pmatrix} \mathbf{0} & -i\sigma_3 \\ i\sigma_3 & \mathbf{0} \end{pmatrix} \quad \tau_5 = \begin{pmatrix} I & \mathbf{0} \\ \mathbf{0} & -I \end{pmatrix}$$

* parameter space

$$\{\lambda, \gamma, \eta, \xi, \zeta\} = S^1 \times S^4$$

Full Solution (J=3/2)

- Quasienergy $\varepsilon_{nj} = \frac{p}{2}\lambda + (-)^n E_{\frac{2-p}{2}\lambda,\gamma,T}$
- Quasieigenstates

$$\theta_\pm = \frac{\xi \pm \zeta}{2}$$

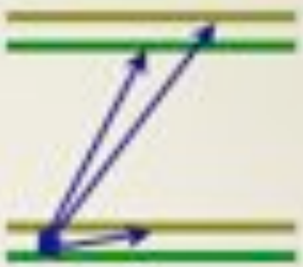
$$q = Q_{\frac{2-p}{2}\lambda,\gamma,T}$$

$$|\Psi_{00}\rangle = \begin{pmatrix} e^{-i\theta_+}\cos q \\ 0 \\ e^{-i\theta_-}\sin\eta\sin q \\ e^{i\theta_-}\cos\eta\sin q \end{pmatrix} \quad |\Psi_{01}\rangle = \begin{pmatrix} 0 \\ e^{i\theta_+}\cos q \\ -e^{-i\theta_-}\cos\eta\sin q \\ e^{i\theta_-}\sin\eta\sin q \end{pmatrix}$$

$$|\Psi_{10}\rangle = \begin{pmatrix} -e^{-i\theta_+}\sin\eta\sin q \\ e^{i\theta_+}\cos\eta\sin q \\ e^{-i\theta_-}\cos q \\ 0 \end{pmatrix} \quad |\Psi_{11}\rangle = \begin{pmatrix} -e^{-i\theta_+}\cos\eta\sin q \\ -e^{i\theta_+}\sin\eta\sin q \\ 0 \\ e^{i\theta_-}\cos q \end{pmatrix}$$

Holonomy All Star

* "Yang-Mills instanton" for S^4 sector



* Mixture of Wilczek-Zee and spiral holonomies for S^1 sector

$$A^{\lambda} = \partial_{\lambda} Q_{\frac{2-p}{2}\lambda,\gamma,T} \begin{bmatrix} 0 & -i\sin\eta I + \cos\eta \Sigma_2 \\ i\sin\eta I + \cos\eta\Sigma_2 & 0 \end{bmatrix}$$

* $$M(C_\lambda) = \cos\frac{(2-p)\pi}{2} - i\sin\frac{(2-p)\pi}{2}\left(\sin\eta \begin{bmatrix} 0 & -iI \\ iI & 0 \end{bmatrix} + \cos\eta \begin{bmatrix} 0 & \Sigma_2 \\ \Sigma_2 & 0 \end{bmatrix}\right)$$

Physical Origin: Diabolical?

* Johanssen-Sjöqvist theorem

  if holonomy on $dS \rightarrow E(\alpha)$: degeneracy on S

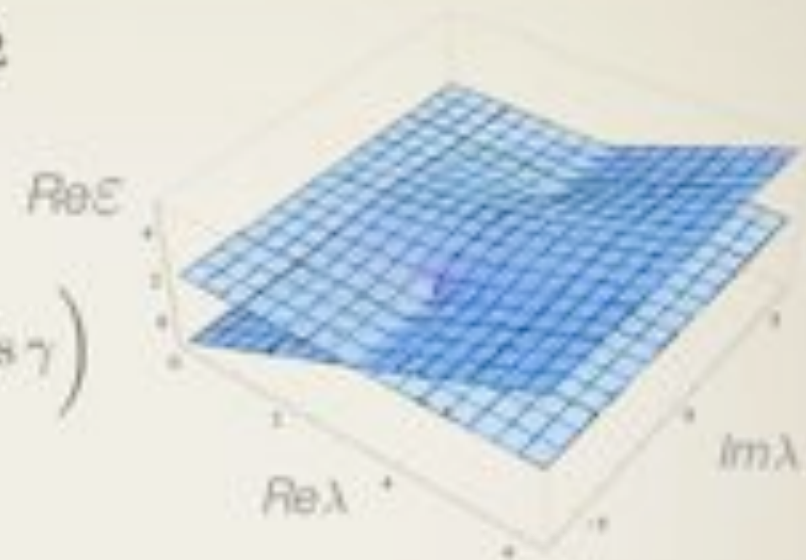
  $\rightarrow A(\alpha)$: singularity on S

* Diabolical point for Berry phase
  on 2-parameter space S^2 etc.

* What about Spiral Holonomy
  - found for parameter space S^1 only
  - hint: complex parameter space singularity?
    --- Exceptional Point

Exceptional Point

* Ex. ; J=1/2 kicked spin, $T=\pi/2$
* Square root singularity for



$$\varepsilon_n(\lambda) = \frac{\lambda}{2} + (-1)^n \arccos\left(-\sin\frac{\lambda}{2}\cos\gamma\right)$$

* Simple pole for $A_\lambda(\lambda)$

$$A_\lambda(\lambda) = \frac{\sin\gamma}{4(\cos\frac{\lambda}{2} - i\sin\frac{\lambda}{2}\sin\gamma)(\cos\frac{\lambda}{2} + i\sin\frac{\lambda}{2}\sin\gamma)}\sigma_2$$

pole $\lambda^*_\pm = \pm 2\arctan\frac{1}{i\sin\gamma}$ residue $\mu_\pm = \pm\frac{1}{4i}$

$$M(C_\lambda) = \mathcal{T}e^{-i\int_0^{2\pi} d\lambda A_\lambda} = e^{i\frac{\pi}{2}\sigma_2} = -i\sigma_2$$

Summary

* Fully gauge invariant formulation of quantum holonomy
* All known quantum holonomies under its fold
* Potentially useful spiral holonomy legitimized
* Illustrative examples including 4 level-system displaying all holonomies mixed together

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