Game Theory formulated on Hilbert Space

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Why Hilbert space game?

- Game theory is a nonlinear dynamical systems theory, gives a mathematical basis of social & biological sciences
- Game theory is based on probability vector and matrix
- Hilbert space game theory?
- Quantum entanglement as classical player correlation?
- Quantum information manipulation as a quantum game?

(Classical) Game matrix

- 2-player n-strategy game specified by

  Payoff matrix $M_{i,j}$ $(i,j = 0..n-1)$

- In symmetric game, opponent's payoff

  is given by $M_{j,i} = M^{+}_{i,j}$

- You (and opponent) try to **maximize payoff** M (M^{+}) by choosing strategy i (j)

Some typical games

prisoner's dilemma

$$M = \begin{pmatrix} -2 & 2 \\ -3 & 0 \end{pmatrix}$$

$M \backslash M^{+}$

you \ opp	confess	deny
confess	-2 \ -2	2 \ -3
deny	-3 \ 2	0 \ 0

monkeys & a fruit tree

$$M = \begin{pmatrix} 0 & 6 \\ 2 & 3 \end{pmatrix}$$

$M \backslash M^{+}$

you \ opp	wait	climb
wait	0 \ 0	6 \ 2
climb	2 \ 6	3 \ 3

Pure Nash equilibria

- **Nash equilibrium** :
  mutual best response
  for "rational" players
  - worse-off by deviating
- In repeated game
  - stable strategy
    after trial & error

you \ opp	confess	deny
confess	-2 \ -2	2 \ -3
deny	-3 \ 2	0 \ 0

Prisoner's Dilemma

you \ opp	wait	climb
wait	0 \ 0	6 \ 2
climb	2 \ 6	3 \ 3

Monkeys&Fruit

Probabilistic strategies

- **Mixed strategy** employable
  Roll dice to spread risk

- **strategy vectors**

$$\mathbf{x} = \begin{pmatrix} 1-x \\ x \end{pmatrix} \quad \mathbf{y} = \begin{pmatrix} 1-y \\ y \end{pmatrix}$$

- Expected **payoffs**

$$\Pi(M, x, y) = \mathbf{x}^\dagger M \mathbf{y}$$

$$\Pi(M^\dagger, x, y) = \mathbf{x}^\dagger M^\dagger \mathbf{y} = \mathbf{y}^\dagger M \mathbf{x} = \Pi(M, y, x)$$

your score matrix $M=\{M_{ij}\}$

you\opp	1-y	y
1-x	M_{00}	M_{01}
x	M_{10}	M_{11}

opp. score matrix $M^{+}=\{M_{ji}\}$

you\opp	1-y	y
1-x	M_{00}	M_{10}
x	M_{01}	M_{11}

your / opponent's payoffs

Mixed Nash equilibrium

- You (x) maximize $\Pi(M, x, y)$
  Opp. (y) maximizes $\Pi(M^\dagger, x, y) = \Pi(M, y, x)$
- Mixed Nash $x^\star$: locally optimal choice of probability
  for symmetric games $\partial_x \Pi(M, x, y)|_{x=y=x^\star} = 0$
- Pure Nash as special cases
  $x^\star = 0$ if $\partial_x \Pi(M, x, y)|_{x=y} < 0$
  $x^\star = 1$ if $\partial_x \Pi(M, x, y)|_{x=y} > 0$

Nash example 1

- Monkeys&fruit tree $M = \begin{pmatrix} 0 & 6 \\ 2 & 3 \end{pmatrix}$ $M^\dagger = \begin{pmatrix} 0 & 2 \\ 6 & 3 \end{pmatrix}$
- Pure Nash : there are two; battle of will

  "hawkish" gets 6 while "dovish" 2

$$\Pi(M, x, y) = 2x(1-y) + 6(1-x)y + 3xy$$

- Random Play Nash $\partial_x \Pi(M, x, y) = 2 - 5y$

  2/5 hawk 3/5 dove $x^\star = 2/5$

  both parties 12/5 $\Pi^\star(M) = \Pi(M, x^\star, x^\star) = 12/5$

Nash example 2

- Prisoner's dilemma $M = \begin{pmatrix} -2 & 2 \\ -3 & 0 \end{pmatrix}$ $M^\dagger = \begin{pmatrix} -2 & -3 \\ 2 & 0 \end{pmatrix}$

$$\Pi(M, x, y) = -2(1-x)(1-y) - 3x(1-y) + 2(1-x)y$$

$$\partial_x \Pi(M, x, y) = -1 - y < 0$$

- Pure Nash $x^\star = 0$: individual rationality

$\Pi^\star(M) = -2$ leading to collective misery

- Optimal (unrealized) result $x^\circ = 1$

trust the other $\Pi^\circ(M) = 0$

Joint-strategy vector

- Joint strategy as **real 4-vector** $\mathbf{z}(x,y) = \mathbf{x} \otimes \mathbf{y}$

$$\mathbf{z}(x,y) = \begin{pmatrix} z_0 \\ z_1 \\ z_2 \\ z_3 \end{pmatrix} = \begin{pmatrix} (1-x)(1-y) \\ (1-x)y \\ x(1-y) \\ xy \end{pmatrix}$$

- Define 4x4 diagonal payoff matrix & full-house vector

$$A = \begin{pmatrix} A_{00} & 0 & 0 & 0 \\ 0 & A_{01} & 0 & 0 \\ 0 & 0 & A_{01} & 0 \\ 0 & 0 & 0 & A_{11} \end{pmatrix},\ A_{ij} = M_{ij}\ ; \quad \mathbf{h} = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix},\ \ \mathbf{h}^\dagger \mathbf{z} = 1$$

- **Payoff** is given by $\Pi(A,x,y) = \mathbf{h}^\dagger A\ \mathbf{z}(x,y)$

Hilbert space strategy

- Think of strategy as Hilbert space vector (Meyer 99, Eisert et al 99)

- Alice's strategy $|\alpha\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$
  Bob's $|\beta\rangle = \beta_0 |0\rangle + \beta_1 |1\rangle$

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \quad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

$$|\alpha_0|^2 + |\alpha_1|^2 = 1$$

$$|\beta_0|^2 + |\beta_1|^2 = 1$$

- Joint strategy : $|\Psi_0\rangle = |\alpha\rangle \otimes |\beta\rangle$

- Payoff as expectation of diagonal matrix *A*

why diagonal?

$$\Pi(A, \alpha, \beta) = \langle \Psi_0 | A | \Psi_0 \rangle$$

$$= \Pi(M, x, y)$$

same as previous result with identification

$$x = |\alpha_1|^2, \; y = |\beta_1|^2$$

Swapping operators

- Define S : $S\left|ij\right\rangle = \left|ji\right\rangle$ $\qquad S\left|\alpha\beta\right\rangle = \left|\beta\alpha\right\rangle$
- Define C : $C\left|ij\right\rangle = \left|\bar{i}\,\bar{j}\right\rangle$

$$\bar{i} = (n-1) - i \qquad \bar{0} = 1,\ \bar{1} = 0$$

- Define T : $T\left|ij\right\rangle = \left|\bar{j}\,\bar{i}\right\rangle$
- D_2 : $T=SC=CS$, $S=CT=TC$, $C=TS=ST$, $S^2=C^2=T^2=1$
- Only two are independent : $S+T-C=1$

Quantum entanglement

- Full Hilbert space strategy in product space U(2)xU(2)
- Correlation operator (2-parameter) : **arbitrator** (3rd person)

$$J(\gamma) = e^{i\gamma_1 S/2} e^{i\gamma_2 T/2} \qquad J(0) = 1$$

- Entangled states span **relevant Hilbert space** uniquely

$$|\alpha\beta;\gamma\rangle = J(\gamma)\,|\alpha\rangle\,|\beta\rangle$$

Cheon & Tsutsui 2006

for example

$$J(\gamma)\,|01\rangle = \cos\frac{\gamma_1}{2}\,|01\rangle + i\sin\frac{\gamma_1}{2}\,|10\rangle$$
$$J(\gamma)\,|00\rangle = \cos\frac{\gamma_2}{2}\,|00\rangle + i\sin\frac{\gamma_2}{2}\,|11\rangle$$

Quantum payoff

Payoff for correlated state

$$\Pi(A, \alpha, \beta; \gamma) = \langle \alpha, \beta; \gamma | \, A \, | \alpha, \beta; \gamma \rangle \; = \langle \alpha, \beta | \, A(\gamma) \, | \alpha, \beta \rangle$$

with $A(\gamma) = J^\dagger(\gamma) A J(\gamma)$

Separation $A(\gamma) = A^{pc}(\gamma) + A^{qi}(\gamma)$

pseudo classical family + quantum interference

$$A^{pc}(\gamma) = \cos^2 \frac{\gamma_1}{2} A + (\cos^2 \frac{\gamma_2}{2} - \cos^2 \frac{\gamma_1}{2}) SAS + \sin^2 \frac{\gamma_2}{2} CAC$$

$$A^{qi}(\gamma) = \frac{i}{2} \sin \gamma_1 (AS - SA) + \frac{i}{2} \sin \gamma_2 (AT - TA)$$

Altruism in classical family

- With $\gamma_2 = 0$, $A^{pc}(\gamma) = \cos^2\frac{\gamma_1}{2}A + \sin^2\frac{\gamma_1}{2}SAS$
- Game with SAS : altruistic dual of game A

$$A \longleftrightarrow SAS \qquad (A_{01} \longleftrightarrow A_{10})$$

$$M \longleftrightarrow M^{\dagger}$$ regards opponents payoff matrix as one's own

- Game with $A^{pc}(\gamma)$: mixture of egoism & altruism (fairness)
- Nash payoff $\Pi^{\star}(A^{pc}(\gamma)) = \Pi(A^{pc}(\gamma), x^{\star}(\gamma), x^{\star}(\gamma))$

  symmetric around $\gamma = \pi/2$: best (worst) payoff there

Pareto efficiency & fairness

- Pareto efficient strategy x°: non-improvable
- For symmetric strategy of symmetric game
  it is given by $\partial_x \Pi(M,x,x)|_{x^\circ} = 0$
- Game with symmetrized A has Pareto efficient Nash

$$\partial_x \Pi(M,x,x) = [\partial_x \Pi(M,x,y) + \partial_x \Pi(M,x,y)]_{x=y}$$
$$= \partial_x \mathbf{x}^\dagger (M + M^\dagger)\mathbf{y}\big|_{x=y}$$

– fairness brings the best results

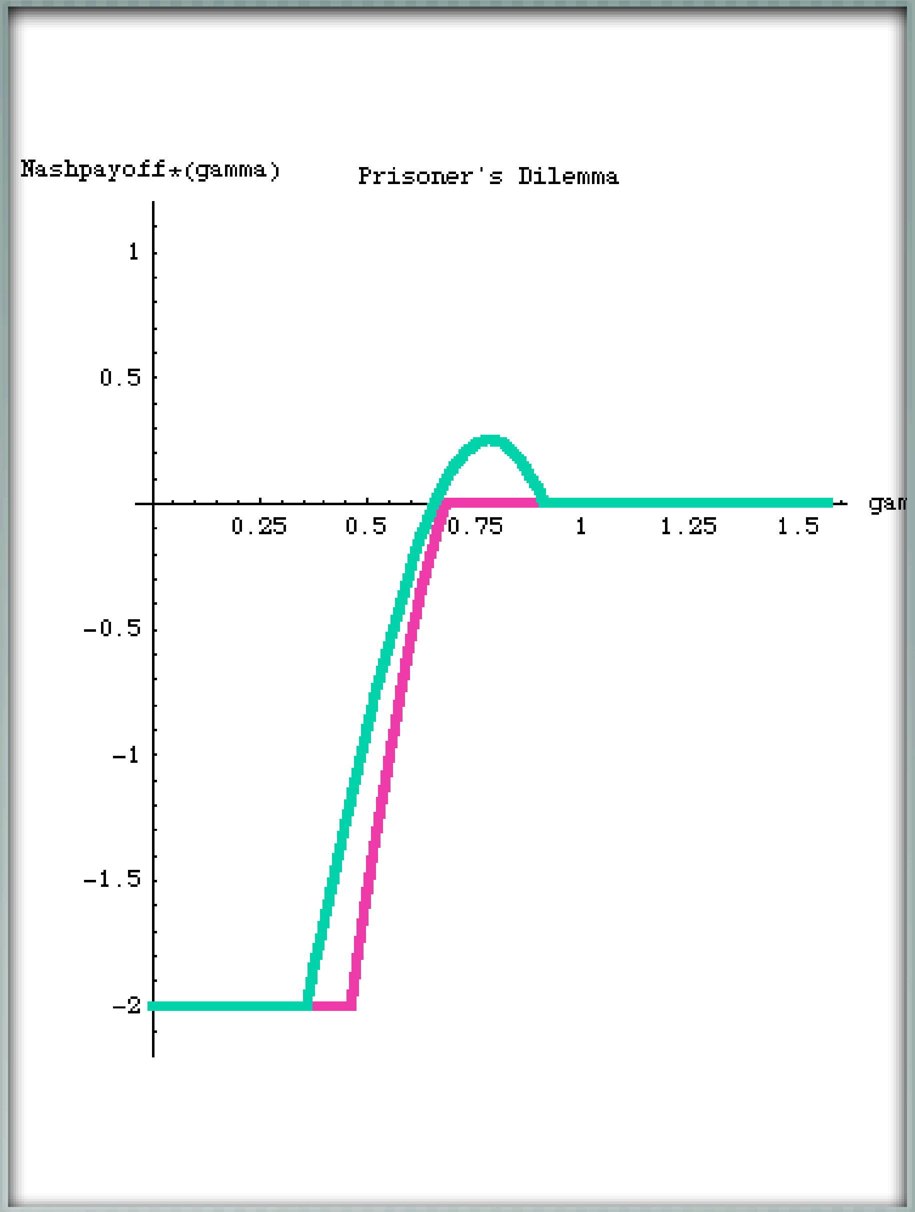
Numerical example 1

- Nash payoff $\Pi^\star(A^{pc}(\gamma))$ of altruistic mixed game
- Prisoner's dilemma

$$M = \begin{pmatrix} -2 & 2 \\ -3 & 0 \end{pmatrix} \quad M = \begin{pmatrix} -2 & 5 \\ -3 & 0 \end{pmatrix}$$

- Nash payoff symmetric aroud $\gamma = \pi/2$:Preto optimal

J. Eisert, M. Wikens and M. Lewenstein, Phys. Rev. Lett. 83 (1999) 3070

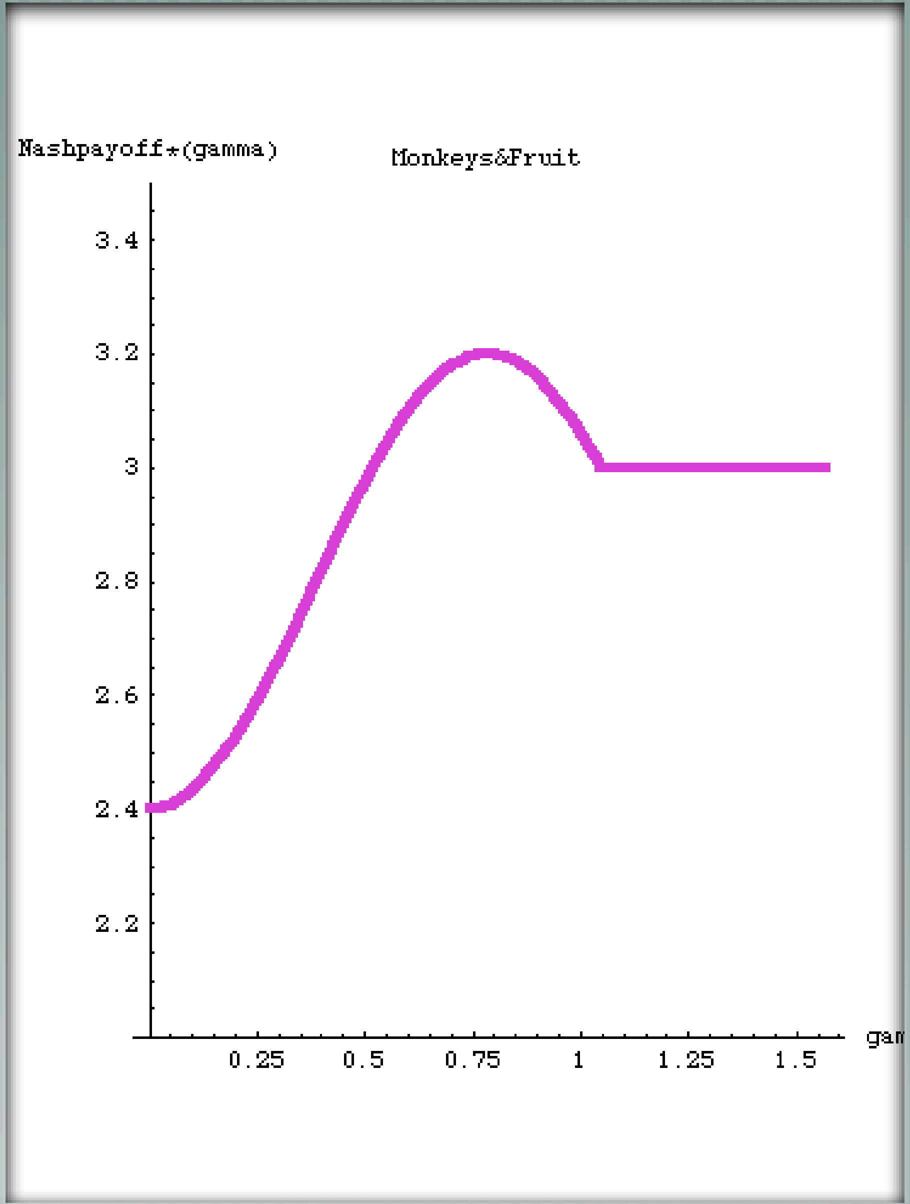


Numerical example 2

- Nash payoff $\Pi^\star(A^{pc}(\gamma))$ of altruistic mixed game
- Monkey and Fruit

$$M = \begin{pmatrix} 0 & 6 \\ 2 & 3 \end{pmatrix}$$

- Nash payoff symmetric around $\gamma = \pi/2$ which is the Preto optimum



Quantum contributions

- $$\langle \alpha\beta | A^{qi}(\gamma) | \alpha\beta \rangle = -a_0 a_1 b_0 b_1 [G_+(\gamma)\sin(\xi + \upsilon) + G_-(\gamma)\sin(\xi - \upsilon)]$$

$$\alpha_0 = a_0,\ \alpha_1 = a_1 e^{i\xi} \qquad G_+(\gamma) = (A_{00} - A_{11})\sin\gamma_2$$

$$\beta_0 = b_0,\ \beta_1 = b_1 e^{i\upsilon} \qquad G_-(\gamma) = (A_{01} - A_{10})\sin\gamma_1$$

- "quantum phase game" turned on with nonzero (ξ, υ)
- Quantum strategy out-performs best classical strategy
- tends to give small correction in many cases

Full solution : edge

- Quantum Nash specified by amplitude $\alpha_1^\star = \sqrt{x^\star} e^{i\xi^\star}$

$$\delta_\alpha \Pi(A, \alpha, \beta)|_{(\alpha^\star, \beta^\star)} = 0 \qquad \delta_\beta \Pi(A, \alpha, \beta)|_{(\alpha^\star, \beta^\star)} = 0$$

- "Edge solution" exists $x^\star = 0 \;\; \text{or} \;\; x^\star = 1$

corresponds to classical pure Nash (can be asymmetric)

$\lvert\alpha^\star, \beta^\star\rangle$	$\lvert 0,0\rangle$	$\lvert 1,1\rangle$	$\lvert 0,1\rangle$	$\lvert 1,0\rangle$
Condition	$H_+ > 0$	$H_- > 0$	$H_+ < 0$	$H_+ < 0$
			$H_- < 0$	$H_- < 0$
$\mathrm{Max}(\Pi_A^\star)$	A_{00}	A_{00}	A_{11}	$A_{01}+A_{10}-A_{11}$
$\mathrm{Max}(\Pi_B^\star)$	A_{00}	A_{00}	$A_{01}+A_{10}-A_{11}$	A_{11}

$$H_+(\gamma) = (A_{00} - A_{11}) \cos\gamma_2 \qquad H_-(\gamma) = (A_{01} - A_{10}) \cos\gamma_1$$

Full solution : non-edge

Phase Nash $\delta_\xi \Pi(\xi, \upsilon)|_{\xi=\upsilon} = 0$

$$\cos 2\xi^\star = G_-(\gamma)/G_+(\gamma)$$

Solution if G+>G-, otherwise uniform random (ξ, υ)

Amplitude Nash $\delta_x \Pi(x, y)|_{x=y} = 0$

$$x^\star = \frac{H_-(\gamma) - \Delta_\gamma}{H_+(\gamma) + H_-(\gamma) - 2\Delta_\gamma}$$

quantum correction $\Delta_\gamma = \sqrt{G_+(\gamma)^2 - G_-(\gamma)^2} \quad (G_+ > G_-)$

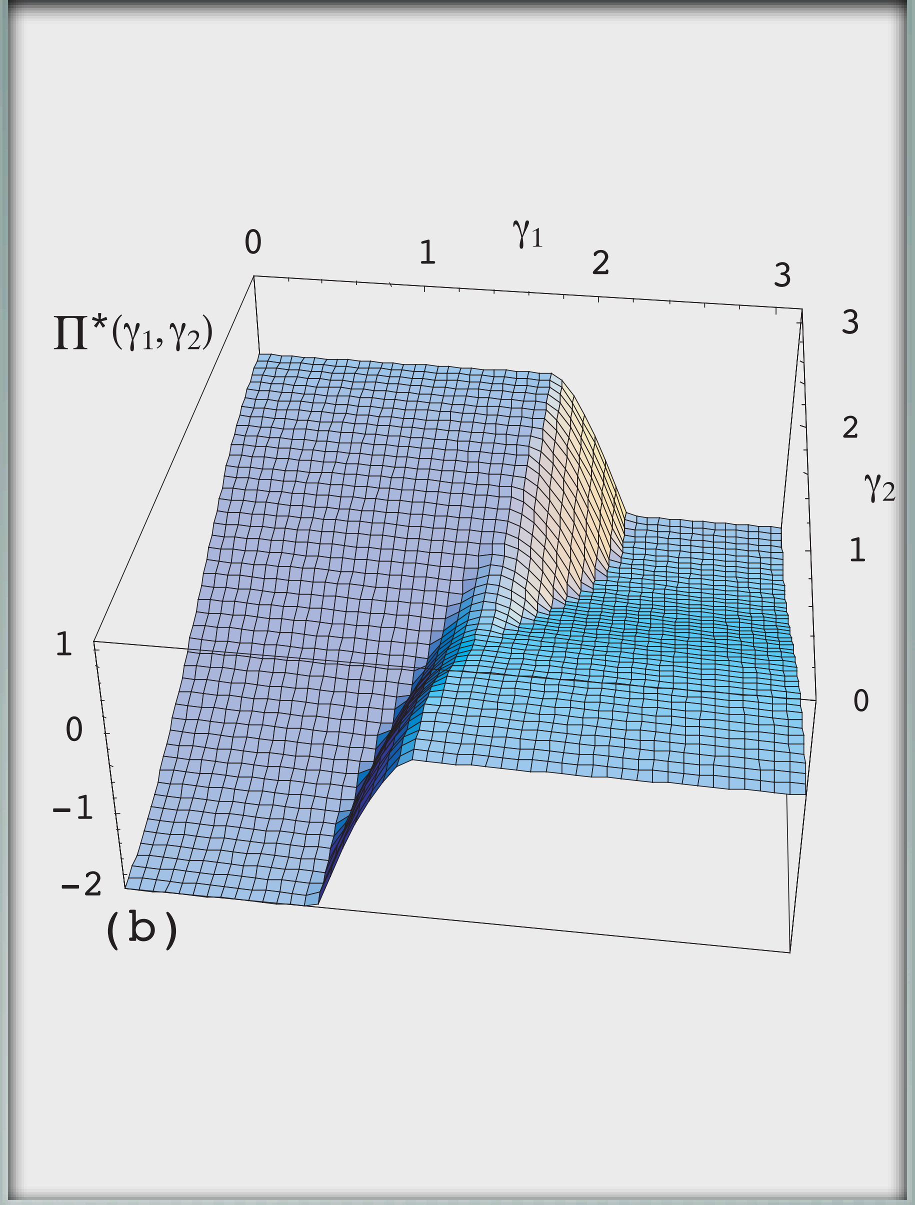
$$\Delta_\gamma = 0 \quad (G_+ < G_-)$$

Numerical example 3

- Quantum strategy in prisoner's dilemma

$$M = \begin{pmatrix} -2 & 2 \\ -3 & 0 \end{pmatrix}$$

- Plot Nash payoff
  - Best result at $\gamma_1 >= \pi/2$

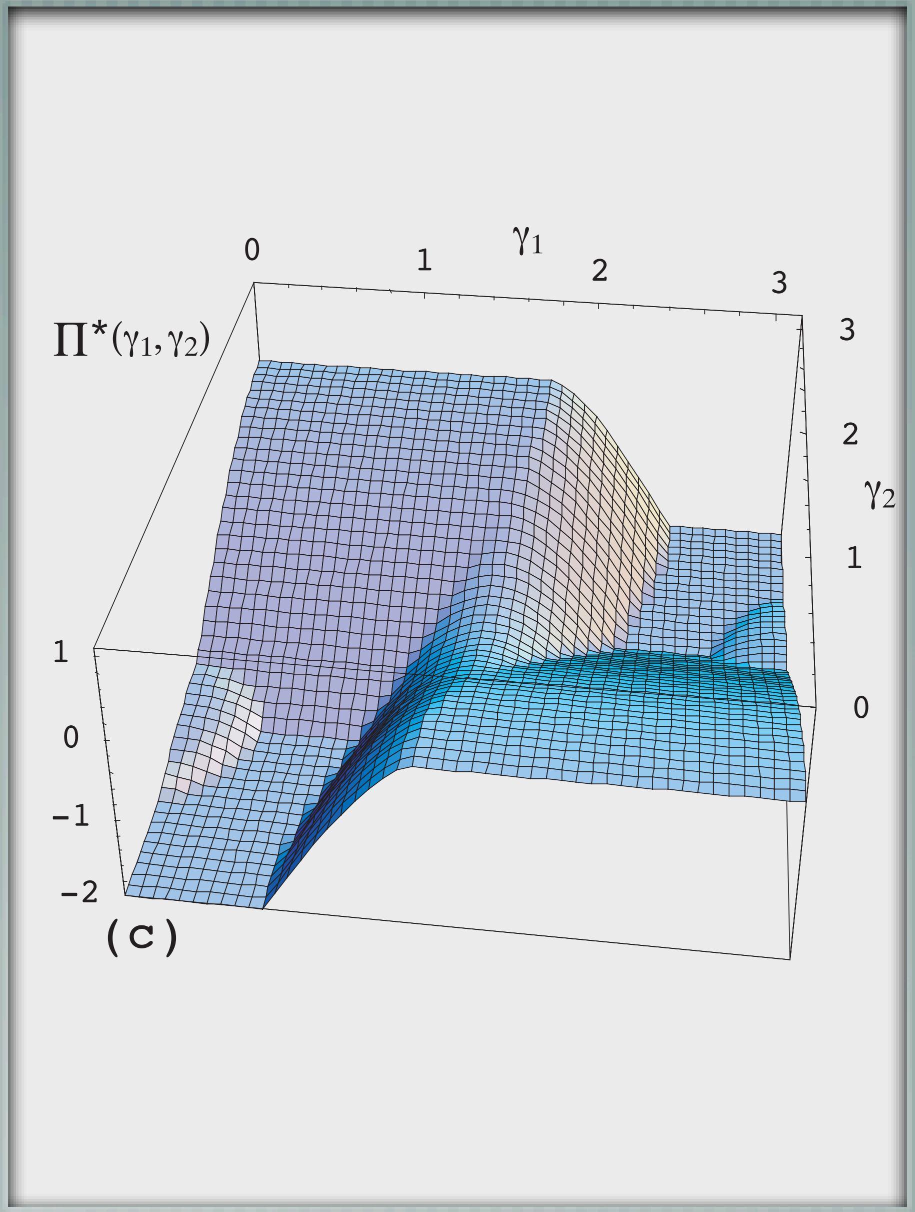


Numerical example 4

- Quantum Payoffs for

$$M = \begin{pmatrix} -2 & 2 \\ -2.8 & 0 \end{pmatrix}$$

- Similar toprevious case apart from appearance of
- “quantum bump”:
  1st ever clear quantum signature



Quantum mixed strategy

- Mixed state expressed by density matrix represents ensemble of quantum states
- Uncorrelated "sum" of $|\Psi_1\rangle$ and $|\Psi_2\rangle$ results in $\Pi(A, c_1, c_2) = c_1 \langle\Psi_1| A |\Psi_1\rangle + c_2 \langle\Psi_2| A |\Psi_2\rangle$
- Ensemble of games
  – can describe role-alternating asymmetric solution
  eg. (x,y) : (0,1)+(1,0), (Π_A, Π_B) : (2,6)+(6,2) $\rightarrow \Pi_{A/B} = 4$

Vista for Hilbert strategies

- Classical pure strategy :
  - Repeated game between two players
- Classical mixed strategy & Quantum pure strategy :
  - Repeated game between two players
  - Game played among players in a group
- Quantum mixed strategy :
  - Ensemble of repeated games/game playing groups

Table of Nash equilibria 1

- Monkey & Fruit $M = \begin{pmatrix} 0 & 6 \\ 2 & 3 \end{pmatrix}$
- classical pure : (x,y)= (0,1), (Π_A, Π_B)=(2,6) and exch.
- classical mixd : x = y = 0.4 , $\Pi_A = \Pi_B$ = 2.4
- quantum pure : x = y = 0.8 , $\Pi_A = \Pi_B$ = 3.2
- quantum mixed : (x,y)= (0,1)+(1,0) , $\Pi_A = \Pi_B$ = 4

Table of Nash equilibria 2

- Prisoner's dilemma $M = \begin{pmatrix} -2 & 2 \\ -3 & 0 \end{pmatrix}$
- classical pure : $x = y = 0$, $\Pi_A = \Pi_B = -2$
- classical mixd : $x = y = 0$, $\Pi_A = \Pi_B = -2$
- quantum pure : $x = y = 1$, $\Pi_A = \Pi_B = 0$
- quantum mixed : $x = y = 1$, $\Pi_A = \Pi_B = 0$

Game with Quantum Objects

- The theory is a full quantum mechanics
  - States as Hilbert space complex vectors
  - Payoffs as Self-adjoint operators
- Applicable to game with real quantum particles
  - Payoff operators not limited to diagonal ones
  - Role of quantum interference & quantum entanglement

Schmidt State Strategy

- Previous CT scheme is insufficient for generic payoffs
- Schmidt state $$|\Psi(\alpha,\beta;\eta)\rangle = U(\alpha)\otimes U(\beta)\,|\Phi(\eta)\rangle$$

Cheon, Ichihara & Tsutsui 2006 $$|\Phi(\eta)\rangle = \cos\frac{\eta_1}{2}\,|00\rangle + e^{i\eta_2}\sin\frac{\eta_1}{2}\,|11\rangle$$

- 0/1 as “spin up/down” $$\sigma_z\,|0\rangle = |0\rangle\,,\quad \sigma_z\,|1\rangle = -\,|1\rangle$$
- Individual strategy: SU(2) Rotation $$U(\alpha) = \begin{pmatrix} \cos\frac{\theta_\alpha}{2} & -e^{-i\varphi_\alpha}\sin\frac{\theta_\alpha}{2} \\ e^{i\varphi_\alpha}\sin\frac{\theta_\alpha}{2} & \cos\frac{\theta_\alpha}{2} \end{pmatrix}$$

Bell's experiment as Game

Purely Quantum Payoff operator

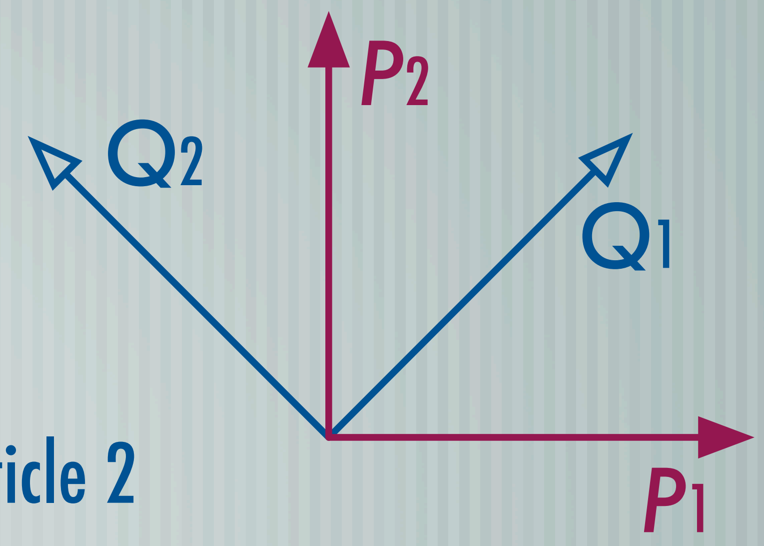
$$A = B = \sqrt{2}\,(\sigma_x \otimes \sigma_x + \sigma_z \otimes \sigma_z)$$

A, B commute with both S and T

Payoff $\Pi = \Pi_A = \Pi_B$

$$\Pi(\alpha, \beta, \eta) = \langle \Psi(\alpha, \beta; \eta)|\, A\, |\Psi(\alpha, \beta; \eta)\rangle$$

Bell's Experiment: maximize



$$P_1 Q_1 - P_2 Q_2 = \Pi(\alpha, \beta, \eta)$$

Alice & Bob control and observe Particle 1 & Particle 2

Tsirelson Bound as Nash

- $\Pi(\alpha, \beta; \eta) = \langle \Phi(\eta)| \left(U^\dagger(\alpha) \otimes U^\dagger(\beta)\, A\, U(\alpha) \otimes U(\beta)\right) |\Phi(\eta)\rangle$
- Equivalent to Alice & Bob controlling directions of *P* and Q, trying to increase Π
- Nash equilibrium under fixed η_1, η_2:

$$\theta^\star_\alpha = \theta^\star_\beta = arbitrary, \;\; \varphi^\star_\alpha = \varphi^\star_\beta = 0$$

- Nash payoff $\Pi^\star(\eta_1, \eta_2) = \sqrt{2}(1 + \sin\eta_1 \cos\eta_2)$

Tsireson's bound at max. entanglement $\eta_1 = \pi/2$, $\eta_2 = 0$

Summary & discussions

- Hilbert space formulation of Game Theory
- Offers unified description of games in
  both microscopic setting and macroscopic setting
- Physical contents of quantum entanglement is exposed as the altruism that often brings about the Pareto Nash
- Research on “truly quantum“games helpful

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