Duality in Quantum Graphs

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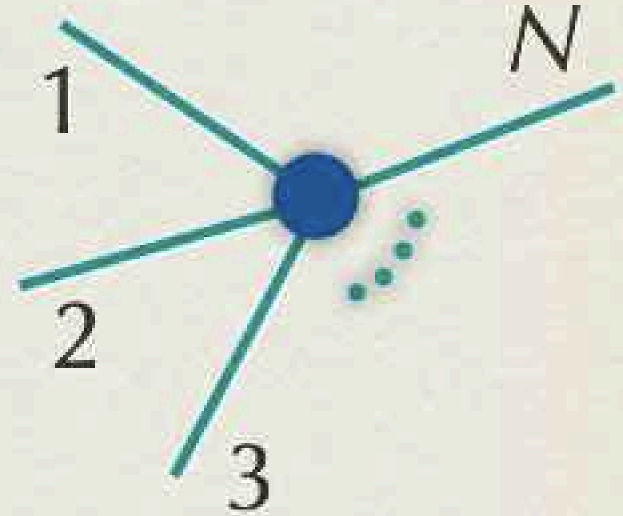
Introduction

- What is quantum graph?
  -- interconnected 1D lines on which quantum particles live
  -- node = singular vertex of degree N

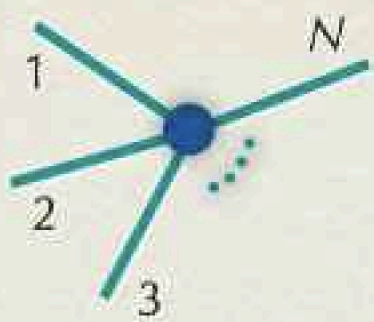


- Why quantum graph?
  -- single electron device with wires
  -- solvable nontrivial quantum mechanics

- What is known?
  -- general mathematical characterization
  -- physical contents analyzed for N=2

General Description

- Boundary vectors



$$\Psi = \begin{pmatrix} \psi_1(0_+) \\ \vdots \\ \psi_N(0_+) \end{pmatrix} \quad \Psi' = \begin{pmatrix} \psi'_1(0_+) \\ \vdots \\ \psi'_N(0_+) \end{pmatrix}$$

- (Self-adjoint extension) Flux conservation $\Psi^\dagger \Psi' = \Psi'^\dagger \Psi$
  -> two N x N matrix A & B ; N^2 parameters

$$A\Psi + B\Psi' = 0$$

$A = I - U \qquad U \in U(N)$

$B = iL_0(I + U)$

(Fulop&Tsutsui '00)

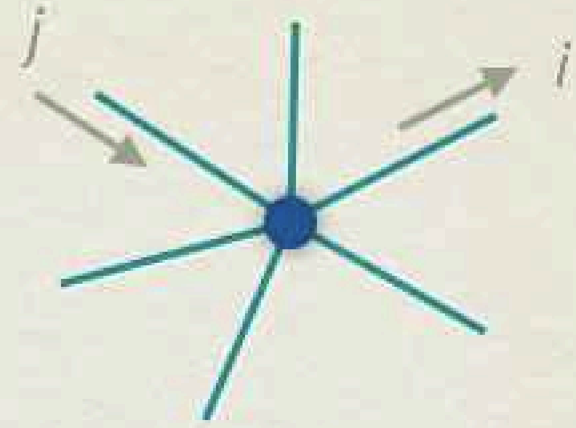
$\mathrm{Rank}(A, B) = N$

$AB^\dagger = BA^\dagger$

(Kostrykin&Schrader '00)

SCATTERING MATRIX

- Scattering for incoming wave at j-th line



$$\psi_i^{(j)}(x_i) = e^{-\mathrm{i}kx_i} + \mathcal{R}_i e^{\mathrm{i}kx_i} \quad (i = j)$$
$$= \mathcal{T}_{ij}\, e^{\mathrm{i}kx_i} \quad (i \neq j)$$

$$\mathcal{S}(k) = \begin{pmatrix} \mathcal{R}_1(k) & \cdots & \mathcal{T}_{1N}(k) \\ \vdots & & \vdots \\ \mathcal{T}_{N1}(k) & \cdots & \mathcal{R}_N(k) \end{pmatrix}$$

$$(\Psi^{(1)} \cdots \Psi^{(N)}) = \mathcal{S}(k) + I$$
$$(\Psi'^{(1)} \cdots \Psi'^{(N)}) = \mathrm{i}k(\mathcal{S}(k) - I)$$

- Scattering Matrix

$$A\Psi + B\Psi' = 0$$

$$\mathcal{S}(k) = -\frac{1}{A + \mathrm{i}kB}(A - \mathrm{i}kB)$$

Rank Reduction

- If $r_B<N$ --> S: $r_B \times r_B$ Hermitian, T: $r_B \times (N-r_B)$

$$\begin{pmatrix} I^{(r_B)} & T \\ 0 & 0 \end{pmatrix} \Psi' = \begin{pmatrix} S & 0 \\ -T^\dagger & I^{(N-r_B)} \end{pmatrix} \Psi$$

$$B\Psi' = -A\Psi$$

r_A=rank(A)

r_B=rank(B)

- If $r_A<N$ --> $\bar{S}$: $r_A \times r_A$ Hermitian, $\bar{T}$: $r_A \times (N-r_A)$

$$\begin{pmatrix} \bar{S} & 0 \\ -\bar{T}^\dagger & I^{(N-r_A)} \end{pmatrix} \Psi' = \begin{pmatrix} I^{(r_A)} & \bar{T} \\ 0 & 0 \end{pmatrix} \Psi$$

- if $r_S < r_A$, $r_{S'} < r_B$ --> further reduction $r_S = r_{S'}$

  $r_S + N = r_A + r_B$

--> "ST" forms, a "standard" form

: basis of physical analysis

GRAPH DUALITY

- System described by $B\Psi' = -A\Psi$

$$\begin{pmatrix} I & T \\ 0 & 0 \end{pmatrix} \Psi' = \begin{pmatrix} S & 0 \\ -T^\dagger & I \end{pmatrix} \Psi$$

and by $A\bar{\Psi}' = -B\bar{\Psi}$

$$\begin{pmatrix} S & 0 \\ -T^\dagger & I \end{pmatrix} \bar{\Psi}' = \begin{pmatrix} I & T \\ 0 & 0 \end{pmatrix} \bar{\Psi}$$

are related to each other

$$\bar{\mathcal{S}}(k) = -\frac{1}{B + \mathrm{i}kA}(B - \mathrm{i}kA)$$

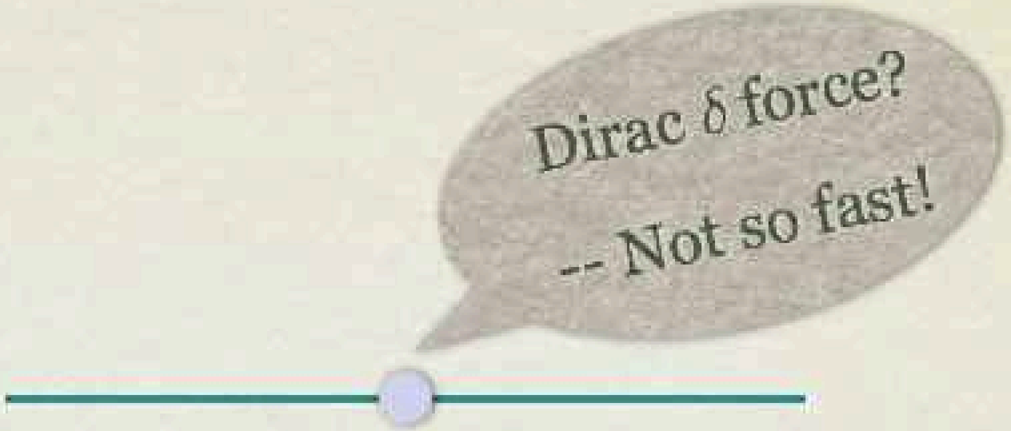
$$= \frac{1}{A + \frac{\mathrm{i}}{-k}B}(A - \frac{\mathrm{i}}{-k}B) = -\mathcal{S}(-1/k)$$

← bound & scattering spectra

N=2 Singular Vertex

- N=2, Point interaction on a line

  N^2=4 parameter family



$$\begin{pmatrix} I^{(r_B)} & T \\ 0 & 0 \end{pmatrix} \Psi' = \begin{pmatrix} S & 0 \\ -T^\dagger & I^{(N-r_B)} \end{pmatrix} \Psi,$$

classify by r_A & r_B

- (r_S=0) r_A= 2 & r_B= 0, r_A= 1 & r_B= 1, r_A= 0 & r_B= 2,

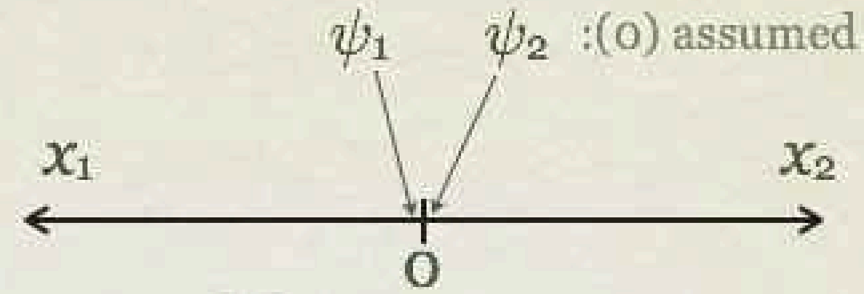
- (r_S =1) r_A= 2 & r_B= 1, r_A= 1 & r_B= 2

- (r_S=2) r_A= 2 & r_B= 2

$r_A + r_B = r_S + 2$

FULOP-TSUTSUI FAMILY

- r_S=0 Boundary $B\Psi'=-A\Psi$



- $r_A=2$, $r_B=0$: $\Psi=0$; $\psi_1=\psi_2=0$: Dirichlet

"classical"

dual partners

- $r_A=0$, $r_B=2$: $\Psi'=0$; $\psi'_1=\psi'_2=0$: Neumann

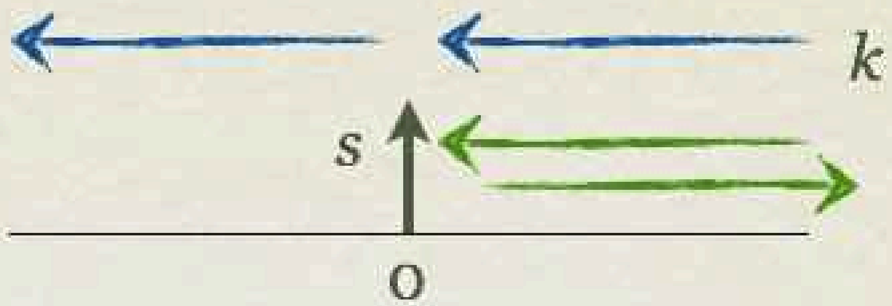
- $r_A=1$, $r_B=1$: $\begin{pmatrix} 1 & t \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1' \\ \psi_2' \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -t^* & 1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

$t^*\psi_1 = \psi_2$ scale invariant complex contact force

$\psi_1' = -t\psi_2'$ (Fulop & Tsutsui '00)

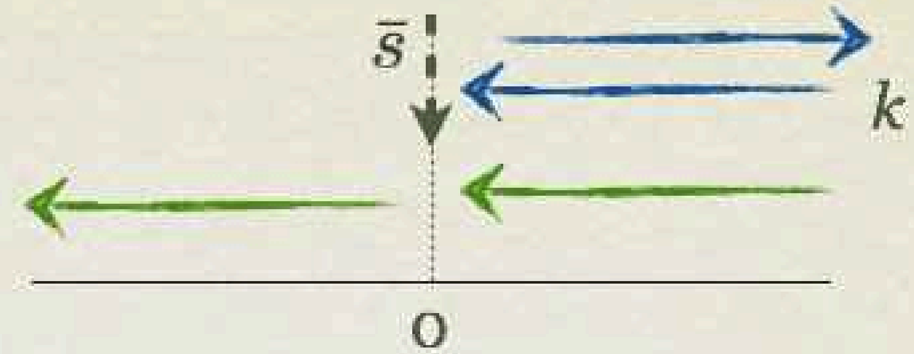
$$\mathcal{T}_{21} = \frac{2t^*}{1+|t|^2}$$

DELTA INTERACTION



- r_S=1 Boundary $B\Psi' = -A\Psi$
- r_A =**2**, r_B =**1** : $\begin{pmatrix} 1 & t \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1' \\ \psi_2' \end{pmatrix} = \begin{pmatrix} s & 0 \\ -t^* & 1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$
- Namely $\psi_1' + t\psi_2' = s\psi_1 = \frac{s}{t^*}\psi_2$ δ-force (t=1) Fermi ('30)
- Parameter s has scale [1/L] : quantum scale anomaly
- High-pass filter: $\mathcal{T}_{12}(k) = \dfrac{2kt}{k(1+t^*t)+\mathrm{i}s}$

DELTA' INTERACTION



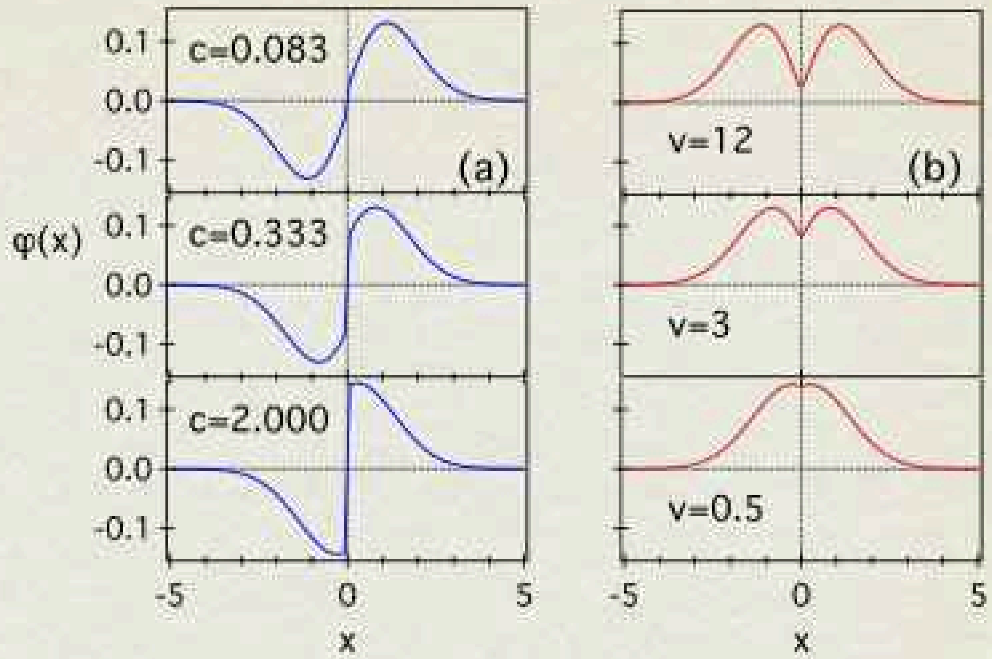
- r_S=1 Boundary $B\Psi'=-A\Psi$
- $r_A=1, r_B=2$: $$\begin{pmatrix} \bar{s} & 0 \\ -\bar{t}^* & 1 \end{pmatrix} \begin{pmatrix} \psi_1' \\ \psi_2' \end{pmatrix} = \begin{pmatrix} 1 & \bar{t} \\ 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix},$$
- Namely $\psi_1 + \bar{t}\psi_2 = \bar{s}\psi_1' = \frac{\bar{s}}{\bar{t}^*}\psi_2'$ δ'-force Seba ('88)
- Parameter $\bar{s}$ has scale [L] : dual to δ-force
- Low-pass filter: $\mathcal{T}_{12}(k) = \dfrac{-2\bar{t}}{(1+\bar{t}^*\bar{t}) - ik\bar{s}}$

acts on antisymmetric ψ_1-ψ_2

contact force for Fermions

Fermi Bose Duality

- Consider symmetric scattering
  -- bosons with
  $\delta\,[v]$ $$\mathcal{S}(k) = \frac{4ik + v}{4ik - v}$$
  -- fermion with
  $\delta'\,[u]$ $$\mathcal{S}(k) = \frac{4ik + 1/u}{4ik - 1/u}$$
- Map to each other with **$vu = 1$**
  (Cheon & Shigehara '99)
  -> 1D δ-bosons/δ'-fermions dual
- useful for Tonks-Girardeau Gas



un dualité plus sexy

DD' Mixture

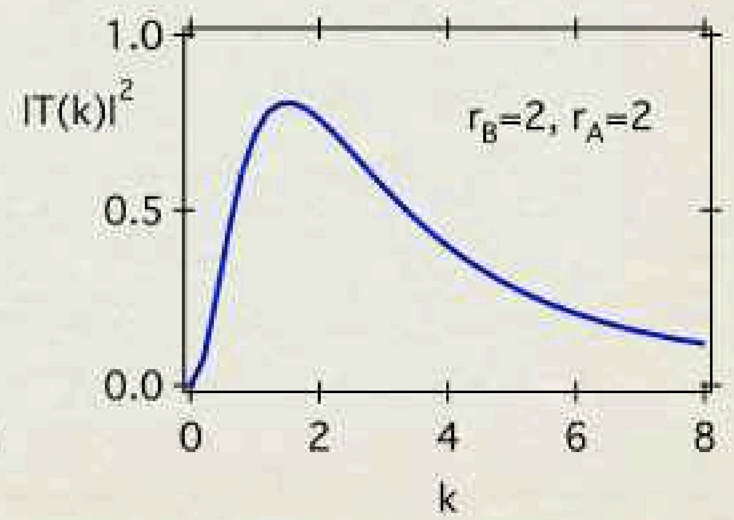
- r_S=2 Boundary $B\Psi' = -A\Psi$

- $r_A = 2, r_B = 2$: $\begin{pmatrix} \psi_1' \\ \psi_2' \end{pmatrix} = \begin{pmatrix} s_{11} & s_{12} \\ s_{12}^* & s_{22} \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}$

- Mixture of δ and δ' interactions

$$\mathcal{T}_{12}(k) = \frac{2ks_{12}}{ik^2 - k\,\mathrm{tr}[S] - \mathrm{i}\det[S]}$$

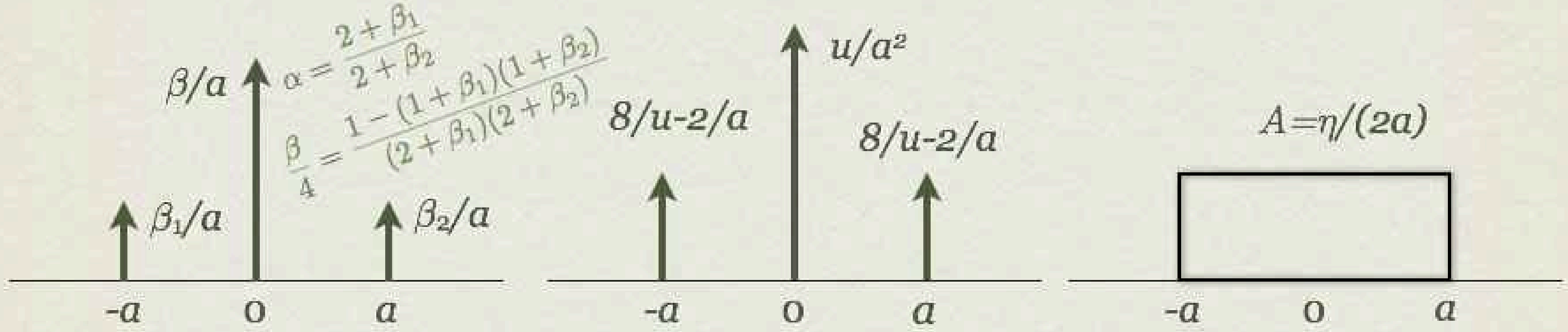
$\mathcal{T}_{12}(k) \to 0$ for both $k \to 0$ and $k \to \infty$



Finite Approximation

- Exotic point forces obtained as rescaled δs with $a \to 0$

$$\begin{pmatrix} \psi'(0_+) \\ \psi(0_+) \end{pmatrix} = \mathcal{V} \begin{pmatrix} \psi'(0_-) \\ \psi(0_-) \end{pmatrix} \qquad \mathcal{V}_\delta(v) = \begin{pmatrix} 1 & \frac{v}{2} \\ 0 & 1 \end{pmatrix} \qquad v : [1/L]$$



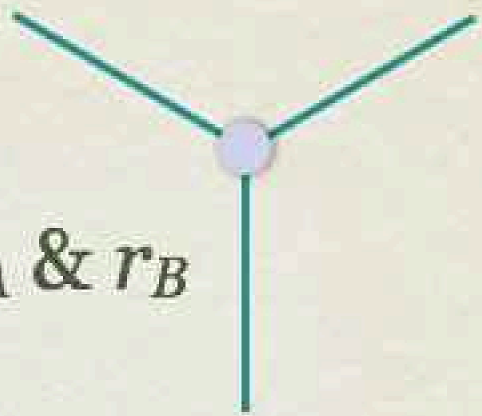
$$\mathcal{V}_{FT}(\alpha) = \begin{pmatrix} \alpha & 0 \\ 0 & \frac{1}{\alpha} \end{pmatrix}, \quad \mathcal{V}_{\delta'}(u) = \begin{pmatrix} 1 & 0 \\ \frac{u}{2} & 1 \end{pmatrix}, \quad \mathcal{V}_A(\eta) = \begin{pmatrix} e^{i\eta} & 0 \\ 0 & e^{i\eta} \end{pmatrix}$$

$\alpha : [1]$ $\qquad$ $u : [L]$

- Cheon & Shigehara '98, Exner, Neidhardt & Zagrebnov '01

N=3 Singular Vertex

- N=3, Y-junction, N^2=9 parameter family



$$\begin{pmatrix} I^{(r_B)} & T \\ 0 & 0 \end{pmatrix} \Psi' = \begin{pmatrix} S & 0 \\ -T^\dagger & I^{(N-r_B)} \end{pmatrix} \Psi,$$

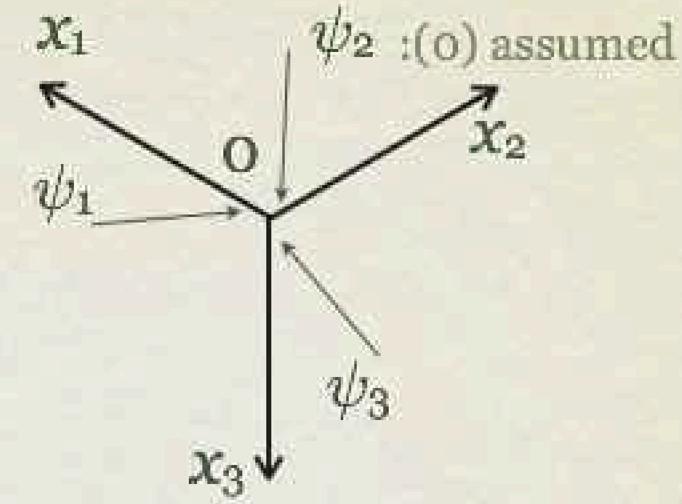
classify by r_A & r_B

- (r_S=0) r_A & r_B : 3 & 0, 2 & 1, 1 & 2, 0 & 3
- (r_S=1) r_A & r_B : 3 & 1, 2 & 2, 1 & 3
- (r_S=2) r_A & r_B : 3 & 2, 2 & 3
- (r_S=3) r_A & r_B : 3 & 3

essential physics observed here

$r_A + r_B = r_S + 3$

FULOP-TSUTSUI FAMILY



- r_S=0 Boundary $B\Psi'=-A\Psi$
- $r_A=3, r_B=0$: $\psi_1=\psi_2=\psi_3=0$: Dirichlet
- $r_A=2, r_B=1$: $\begin{pmatrix}1 & t_2 & t_3\\ 0 & 0 & 0\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\psi_1'\\ \psi_2'\\ \psi_3'\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ -t_2^* & 1 & 0\\ -t_3^* & 0 & 1\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}$

  3bdy FT(1)
- $r_A=1, r_B=2$: $\begin{pmatrix}1 & 0 & \bar{t}_1\\ 0 & 1 & \bar{t}_2\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\psi_1'\\ \psi_2'\\ \psi_3'\end{pmatrix}=\begin{pmatrix}0 & 0 & 0\\ 0 & 0 & 0\\ -\bar{t}_1^* & -\bar{t}_2^* & 1\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}$

  3bdy FT(2)
- $r_A=0, r_B=3$: $\psi'_1=\psi'_2=\psi'_3=0$: Neumann

FREE' CONNECTION

- Two types of Fulop-Tsutsui for N=3 and beyond
  <-- two types of "free" connection conditions (t_i=1)
  Free $\psi_1 = \psi_2 = \psi_3,\ \psi'_1 + \psi'_2 + \psi'_3 = 0$
  $$\mathcal{T}_{jk} = 2/3,\ \mathcal{R}_j = 1/3$$
  Free' $\psi'_1 = \psi'_2 = \psi'_3,\ \psi_1 + \psi_2 + \psi_3 = 0$
  $$\mathcal{T}_{jk} =-2/3,\ \mathcal{R}_j =-1/3$$
  -- Dual (i*k* <-> 1/i*k*), and *k*-independent

- free connection breaks Ψ-Ψ' symmetry
  -- free' usable for grid-regularization ?
  -- more surprises for *N*>3 ?

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  Free' $\psi'_1 = \psi'_2 = \psi'_3, \; \psi_1 + \psi_2 + \psi_3 = 0$
  $$\mathcal{T}_{jk} = -2/3, \; \mathcal{R}_j = -1/3$$
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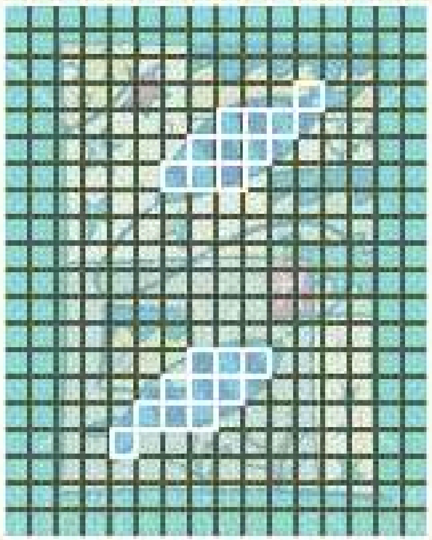
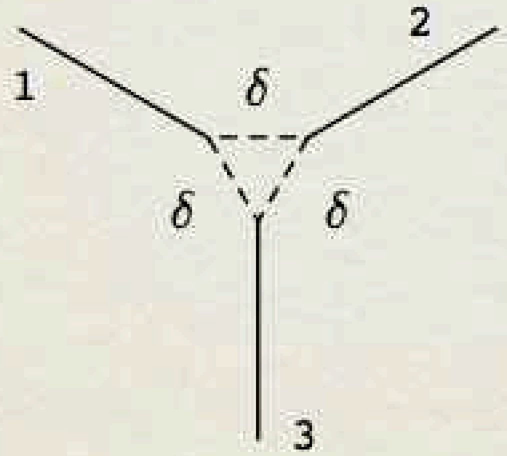


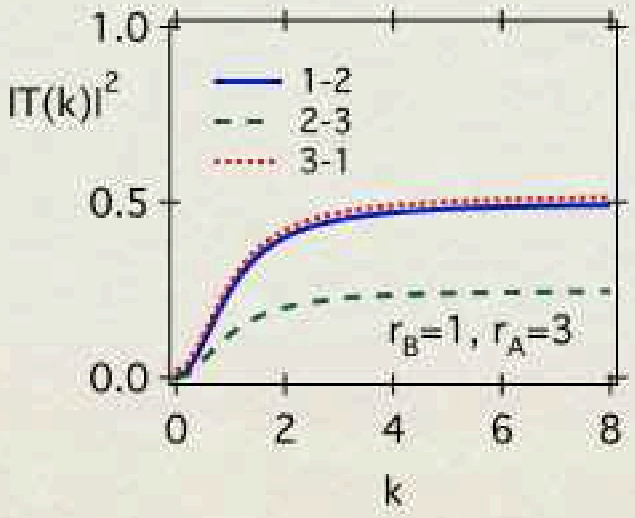
- free connection breaks Ψ-Ψ' symmetry
  -- free' usable for grid-regularization ?
  -- more surprises for *N*>3 ?

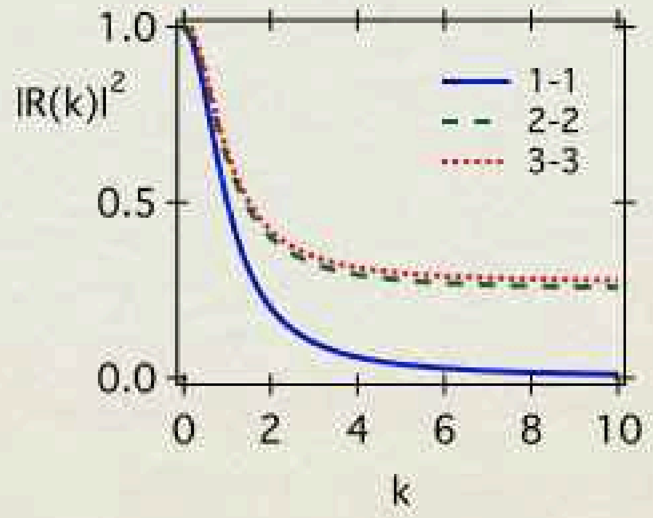
Delta Type

- r_S=1 Boundary $B\Psi' = -A\Psi$
- r_A =3, r_B =1 : $\begin{pmatrix} 1 & t_2 & t_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1' \\ \psi_2' \\ \psi_3' \end{pmatrix} = \begin{pmatrix} s & 0 & 0 \\ -t_2^* & 1 & 0 \\ -t_3^* & 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$
- With t_2=t_3=1,

  "generalized δ"-type $\psi_1' + \psi_2' + \psi_3' = s\psi_1 = s\psi_2 = s\psi_3$







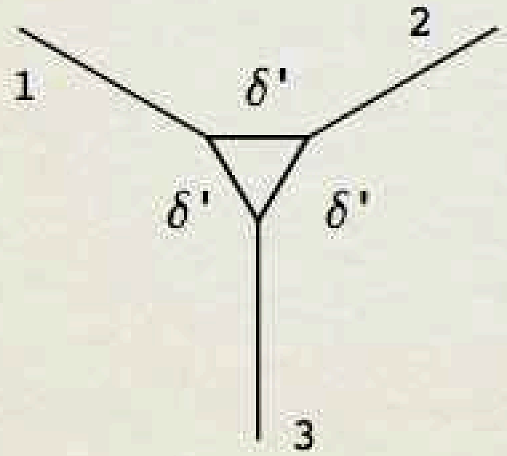
DELTA' TYPE

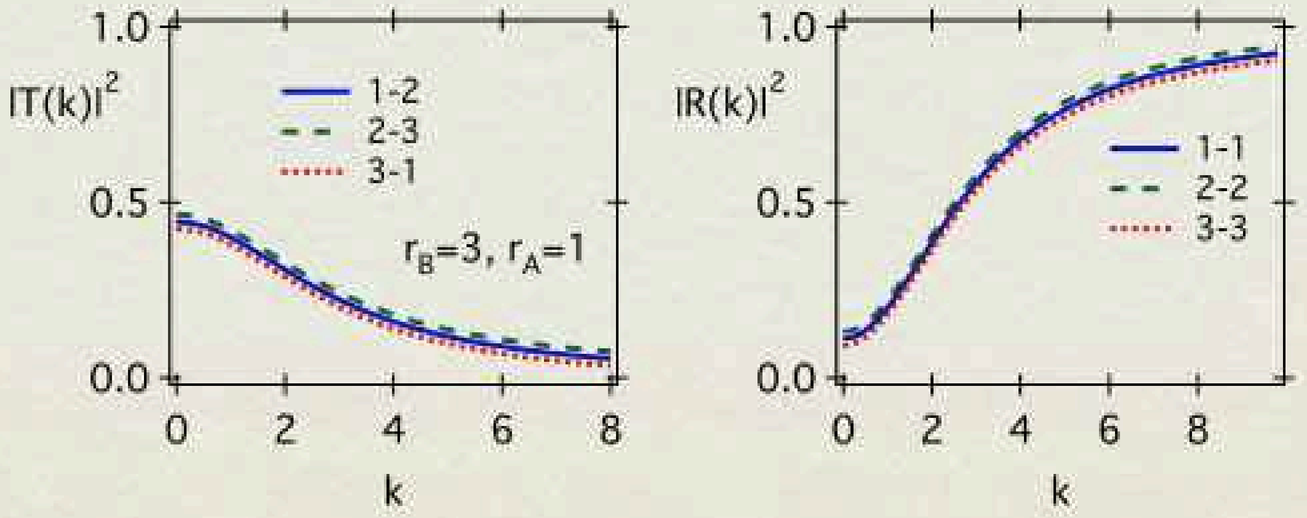
- r_S=1 Boundary $B\Psi'$=-$A\Psi$
- r_A =1, r_B =3 :

$$\begin{pmatrix} s & 0 & 0 \\ -t_2^* & 1 & 0 \\ -t_3^* & 0 & 1 \end{pmatrix} \begin{pmatrix} \psi_1' \\ \psi_2' \\ \psi_3' \end{pmatrix} = \begin{pmatrix} 1 & t_2 & t_3 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

- With t_2=t_3=1,

  generalized δ'-type $\psi_1 + \psi_2 + \psi_3 = s\psi_1' = s\psi_2' = s\psi_3'$





Delta-Delta' Mix Type

- r_S=1 r_A =**2**, r_B =**2**

  $B\Psi$'=-$A\Psi$ can take two dual forms

- $$\begin{pmatrix} 1 & 0 & t_1 \\ 0 & 1 & t_2 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi'_1 \\ \psi'_2 \\ \psi'_3 \end{pmatrix} = \begin{pmatrix} s & cs & 0 \\ c^* s & c^* cs & 0 \\ -t_1^* & -t_2^* & 1 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_2 \\ \psi_3 \end{pmatrix}$$

- $$\begin{pmatrix} \bar{s} & \bar{c}\bar{s} & 0 \\ \bar{c}^* \bar{s} & \bar{c}^* \bar{c}\bar{s} & 0 \\ -\bar{t}_1^* & -\bar{t}_3^* & 1 \end{pmatrix} \begin{pmatrix} \psi'_1 \\ \psi'_3 \\ \psi'_2 \end{pmatrix} = \begin{pmatrix} 1 & 0 & \bar{t}_1 \\ 0 & 1 & \bar{t}_3 \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_3 \\ \psi_2 \end{pmatrix}$$

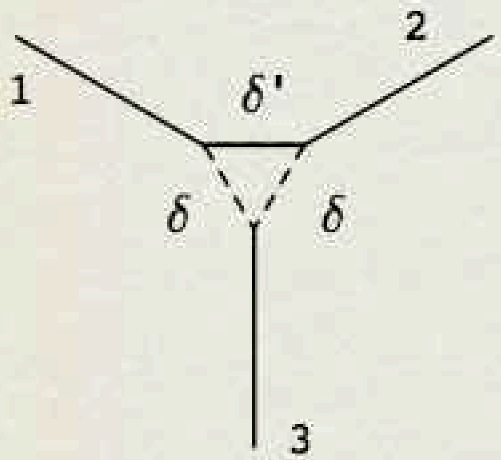
  where $\bar{s} = 1/s,\ \bar{c} = t_1,\ \bar{t}_1 = c,\ \bar{t}_3 = ct_1^* - t_2^*$

DDD' TYPE

- r_S=1 $\underline{r_A = \mathbf{2}, r_B = \mathbf{2}}$ $: \ \bar{t}_3 = 0$ $\begin{pmatrix} \bar{s} & \bar{c}\bar{s} & 0 \\ \bar{c}^*\bar{s} & \bar{c}^*\bar{c}\bar{s} & 0 \\ -\bar{t}_1^* & \mathbf{0} & 1 \end{pmatrix} \begin{pmatrix} \psi_1' \\ \psi_3' \\ \psi_2' \end{pmatrix} = \begin{pmatrix} 1 & 0 & \bar{t}_1 \\ 0 & 1 & \mathbf{0} \\ 0 & 0 & 0 \end{pmatrix} \begin{pmatrix} \psi_1 \\ \psi_3 \\ \psi_2 \end{pmatrix}$

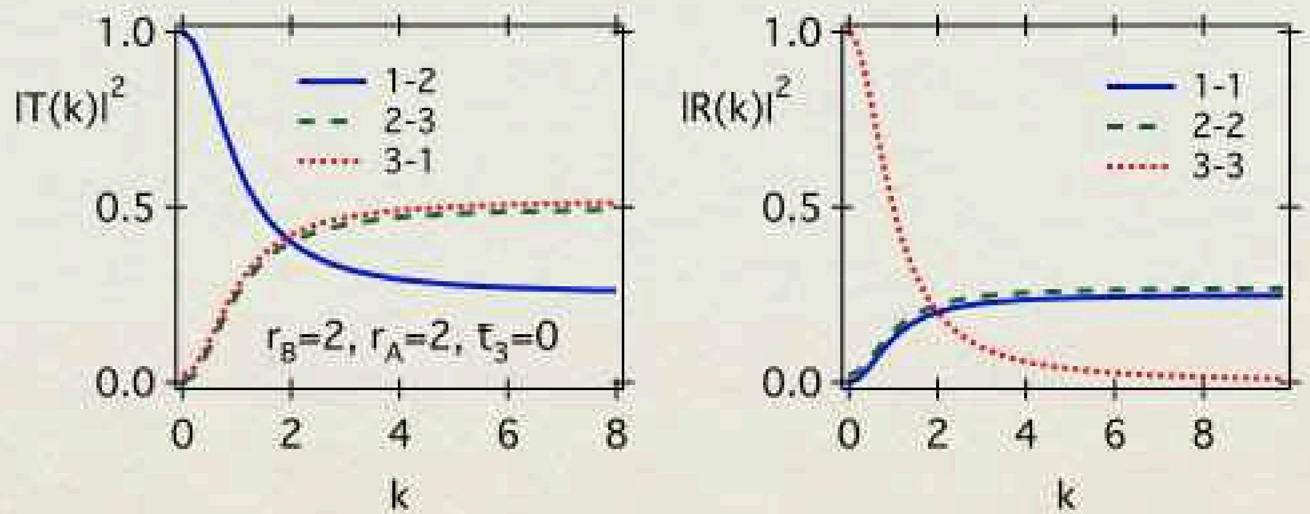
- δ-δ-δ'-type

$$\frac{1}{t_1}\psi_1' + \psi_3' = \frac{1}{t_2}\psi_2' + \psi_3' = \frac{s}{t_1^* t_1}\psi_3$$



$$\psi_3 = t_1^*\psi_1 + t_2^*\psi_2 \qquad \frac{1}{t_1}\psi_1' = \frac{1}{t_2}\psi_2'$$

Branch Filtering <--



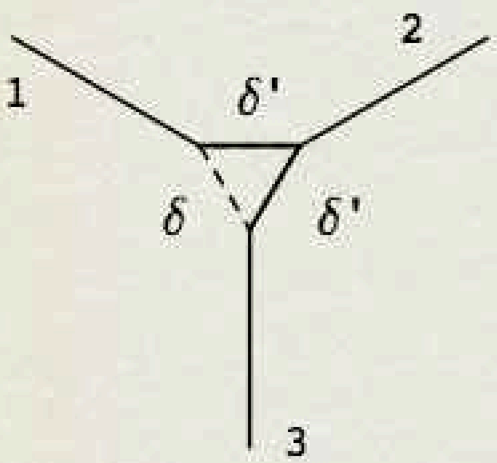
DD'D' Type

- $r_S=1$ $\underline{r_A=\mathbf{2},\ r_B=\mathbf{2}}$: $t_2=0$

$$\begin{pmatrix}1 & 0 & t_1\\ 0 & 1 & \mathbf{0}\\ 0 & 0 & 0\end{pmatrix}\begin{pmatrix}\psi_1'\\ \psi_2'\\ \psi_3'\end{pmatrix}=\begin{pmatrix}s & cs & 0\\ c^*s & c^*cs & 0\\ -t_1^* & \mathbf{0} & 1\end{pmatrix}\begin{pmatrix}\psi_1\\ \psi_2\\ \psi_3\end{pmatrix}$$

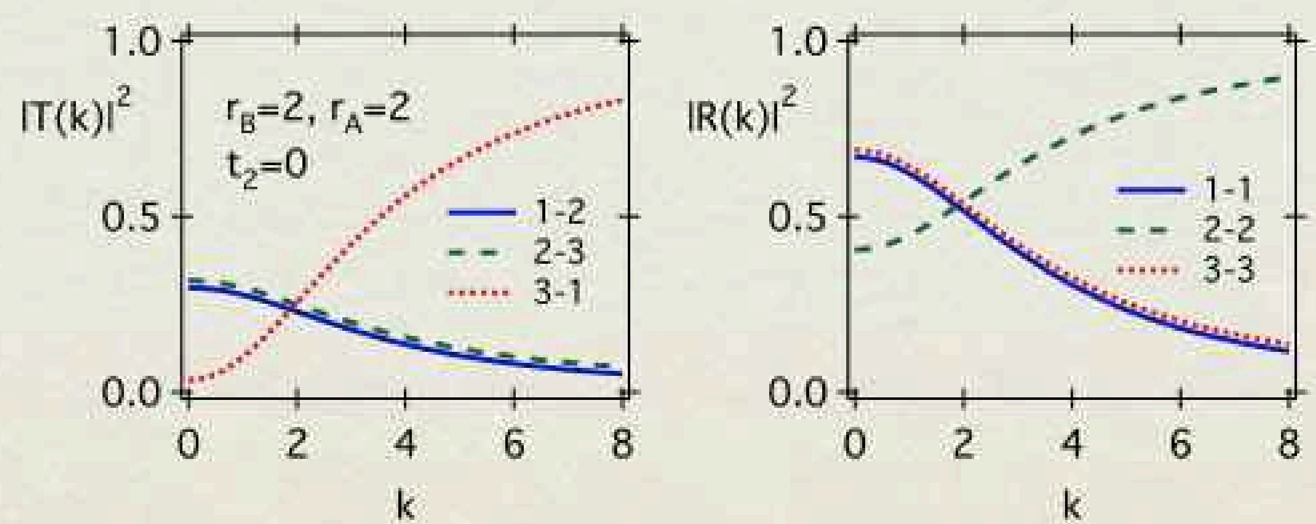
- δ-δ-δ'-type

$$\frac{1}{c}\psi_1+\psi_2=\frac{1}{ct_1^*}\psi_3+\psi_2=\frac{1}{sc^*c}\psi_2'$$



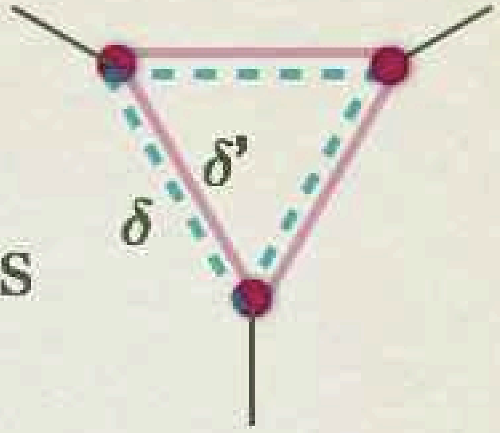
$$\psi_2'=c^*\psi_1'+c^*t_1\psi_3' \qquad t_1^*\psi_1=\psi_3$$

Branch Filtering <--



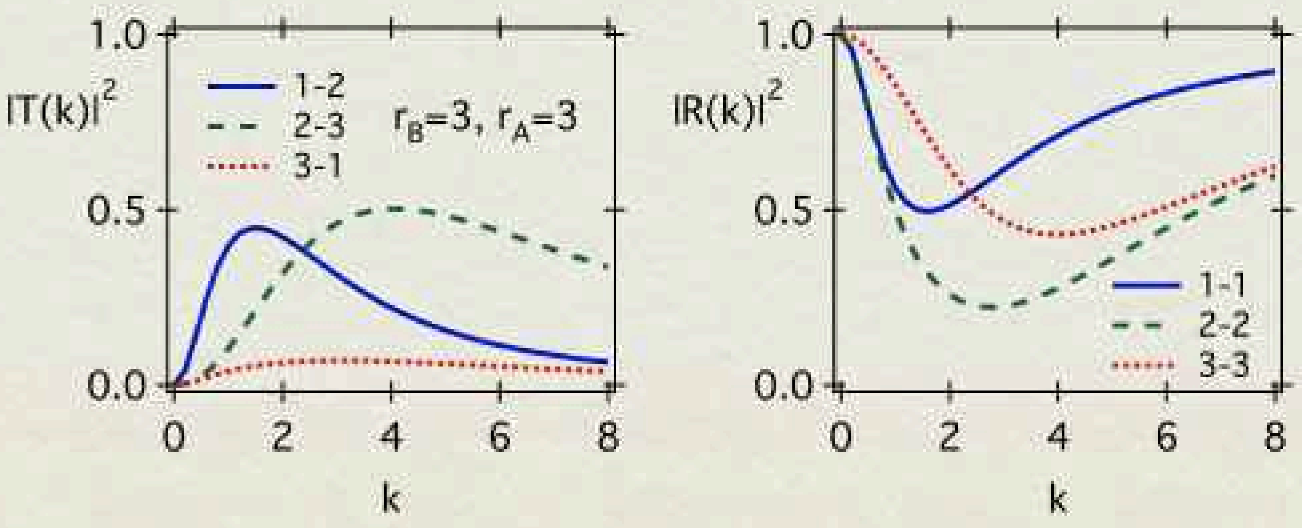
MIXTURE OF D&D' TYPES

- r_S=2 $\underline{r_A = 3, r_B = 2}$ and $\underline{r_A = 2, r_B = 3}$
  r_S=3 $\underline{r_A = 3, r_B = 3}$: generic mixture of δ and δ' types for all pairs



$$\mathcal{T}_{ij}(k) = \frac{-2ik^2 s_{ij} + 2k\,\det[\mathcal{S}_{ji}]}{k^3 + ik^2\mathrm{tr}[S] - k\sum_i \det[\mathcal{S}_{ii}] - i\det[S]}$$

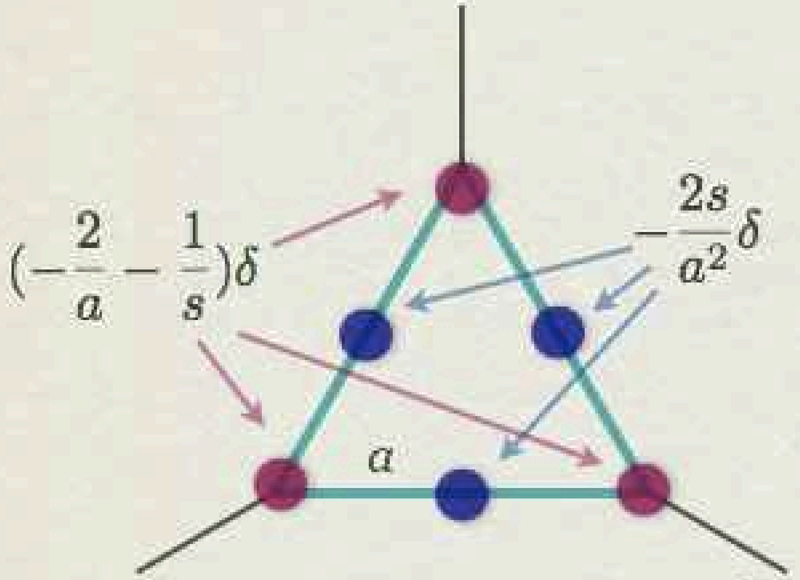
- Example



FINITE APPROXIMATION

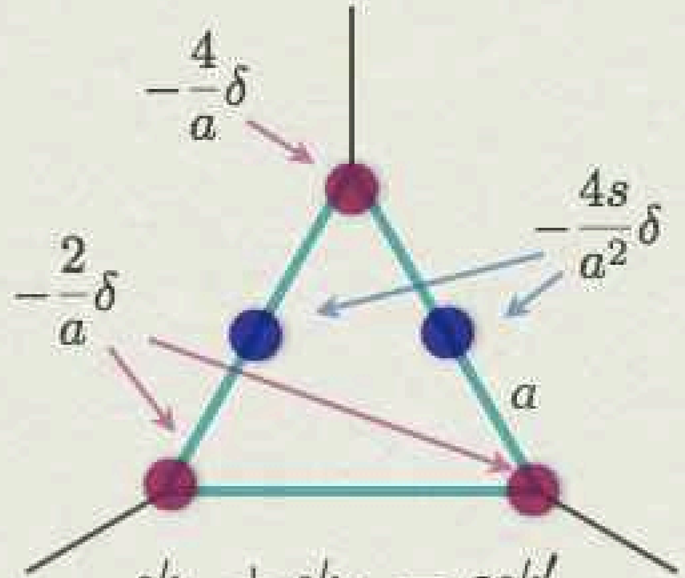
- Exotic singular vertex can be approximated by ...

$\delta'\delta'\delta'$



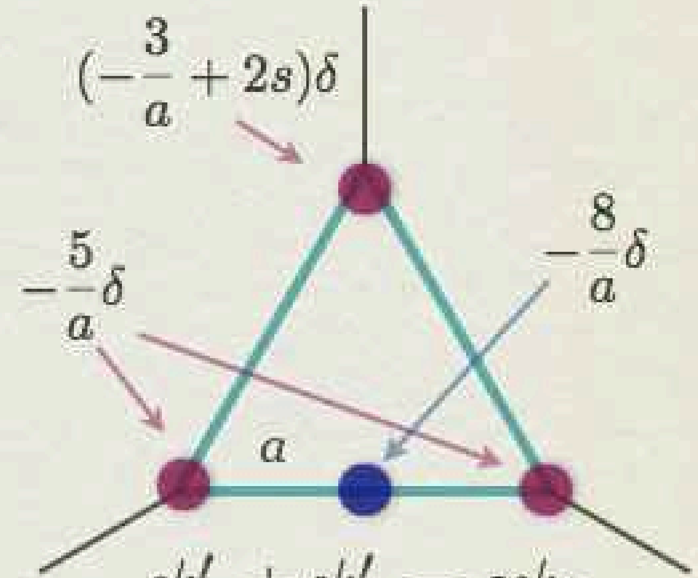
$$\psi_1 + \psi_2 + \psi_3 = s\psi'_j$$
$$\psi'_1 = \psi'_2 = \psi'_3$$

$\delta'\delta'\delta$



$$\psi_1 + \psi_3 = s\psi'_3$$
$$\psi'_3 = \psi'_1 + \psi'_2$$
$$\psi_1 = \psi_2$$

$\delta\delta\delta'$



$$\psi'_1 + \psi'_3 = s\psi_3$$
$$\psi_3 = \psi_1 + \psi_2$$
$$\psi'_1 = \psi'_2$$

- Norm convergence

3+3+3=9=3^2

General Structure

- General characterization of singular vertex
  -- Scattering property "pairwise tunable"
  -- Each pair of lines (j, k): mix of δ and δ'

$$\begin{pmatrix} I^{(r_B)} & T \\ 0 & 0 \end{pmatrix} \Psi' = \begin{pmatrix} S & 0 \\ -T^\dagger & I^{(N-r_B)} \end{pmatrix} \Psi, \quad \begin{pmatrix} \bar{S} & 0 \\ -\bar{T}^\dagger & I^{(N-r_A)} \end{pmatrix} \Psi' = \begin{pmatrix} I^{(r_A)} & \bar{T} \\ 0 & 0 \end{pmatrix} \Psi$$

Key quantities: rank(A), rank(B)

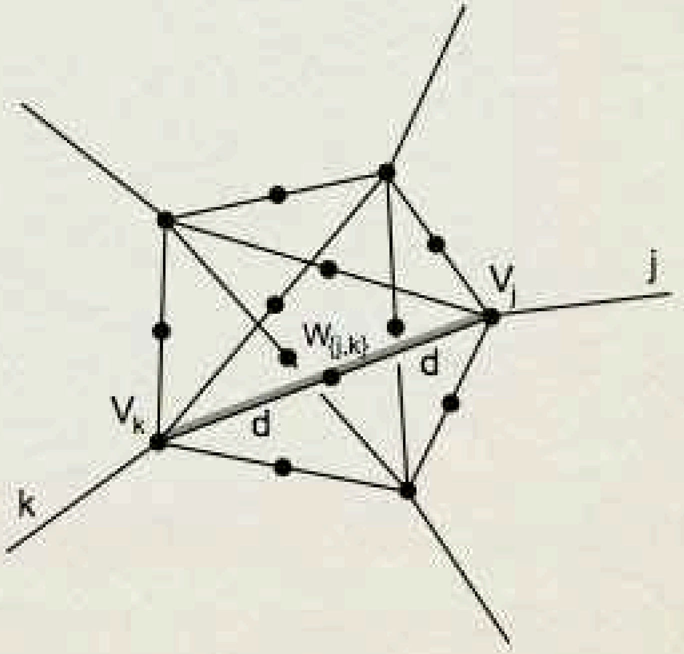
- Another form

$$\begin{pmatrix} I^{(r_S)} & 0 & P \\ R & I^{(N-r_A)} & Q+RP \\ 0 & 0 & 0 \end{pmatrix} \Psi' = \begin{pmatrix} S & -SR^\dagger & 0 \\ 0 & 0 & 0 \\ -P^\dagger & -Q^\dagger & I^{(N-r_B)} \end{pmatrix} \Psi$$

- Number of parameters $= N^2 - (N-r_A)^2 - (N-r_B)^2$

Finite Approximation

- Finite approximation scheme for general singular vertex
  - -- cut the node, connect all pairs with lines (j, k) length a
  - -- place N ds $[v_j]$ at new nodes
  - -- place $N(N\text{-}1)/2$ ds $[w_{jk}]$ at the center of (j, k)
  - -- place $N(N\text{-}1)/2$ vector potentials $[A_{jk}]$ on (j, k)
  - -- let $v_j = \gamma_j + \beta_j/a$, $A_{ij} = \eta/a$
  - -- let $w_{jk} = \beta_{jk}/a + \zeta_{jk}/a^2$
  - -- balance tune parameters as $a \to 0$



$N+N(N\text{-}1)/2+N(N\text{-}1)/2=N^2$

- Norm convergence proved

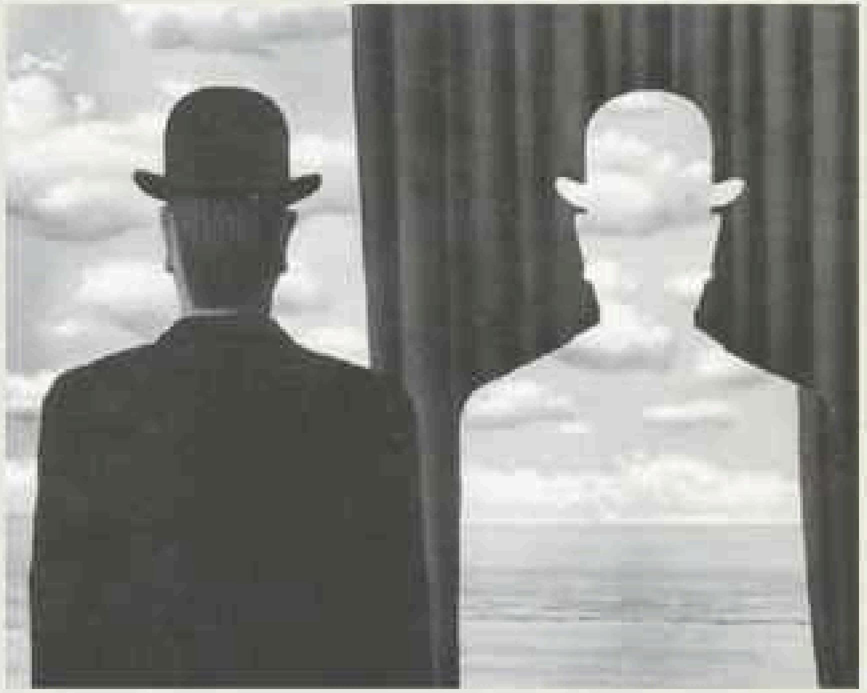
FINAL REMARKS

- Y-junction branching filter as single electron transistor
  <- electric field to legs

- Root cause of graph duality:
  Ψ-Ψ' symmetry of Helmholz eq.

  $$1/(ik)\Psi'' = ik\Psi$$

  split into dual partners by
  geometric structure and by
  scale anomaly of singular vertex



- Minute 1D graph-like structures : duality in spectra ...

CONCLUSION

- Singular quantum vertex of degree *N* analyzed
  -- rank reduction of boundary matrix reveals physical contents
  -- scale anomaly and duality clarified
- Finite approximation scheme constructed
- Prototype of quantum wire devices
- Advanced quantum mechanics in elementary settings

References

- A work in collaboration with Pavel Exner & Ondrej Turek

- T. Cheon, P. Exner, O. Turek,
  Approximation of a General Singular Vertex Coupling in Quantum Graph, arXiv: 0908.2679 (quant-ph) (36p)

- T. Cheon, P. Exner, O. Turek,
  Spectral Filtering in Quantum Y-Junction,
  arXiv: 0908.3717 (quant-ph) (7p)

- See http://www.kochi-tech.ac.jp/~cheon/