Quantum Abacus

Qubit Realization with Quantum Point Interactions

TAKSU CHEON
(KOCHI TECH)

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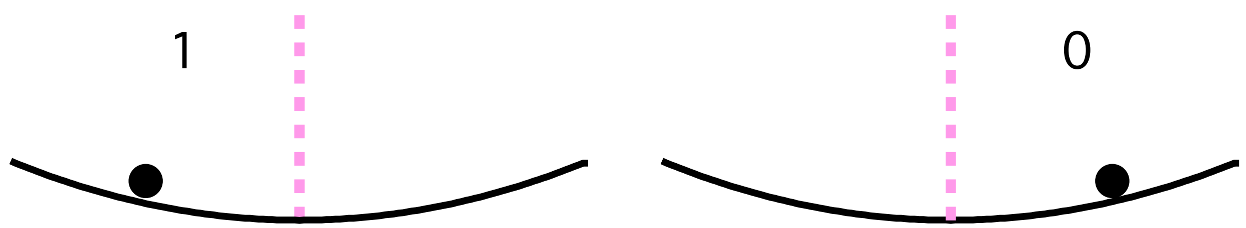
Introduction

- Theoretical design of semiclassically robust qubit
- Self-driven classical abacus based on "locational" bit controlled by gate
- Quantum abacus based on recurring quantum states, control by quantum gate that is described by U(2) point interaction

Classical abacus

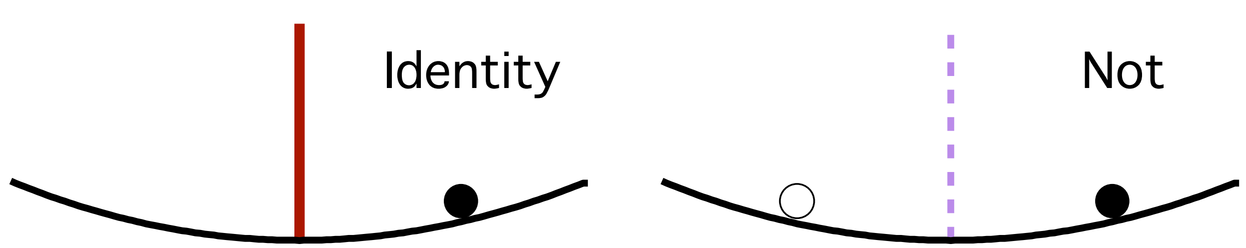
- hard ball in harmonic oscillator stores a BIT as right/left location



$$V = \frac{m\omega^2}{2}x^2$$

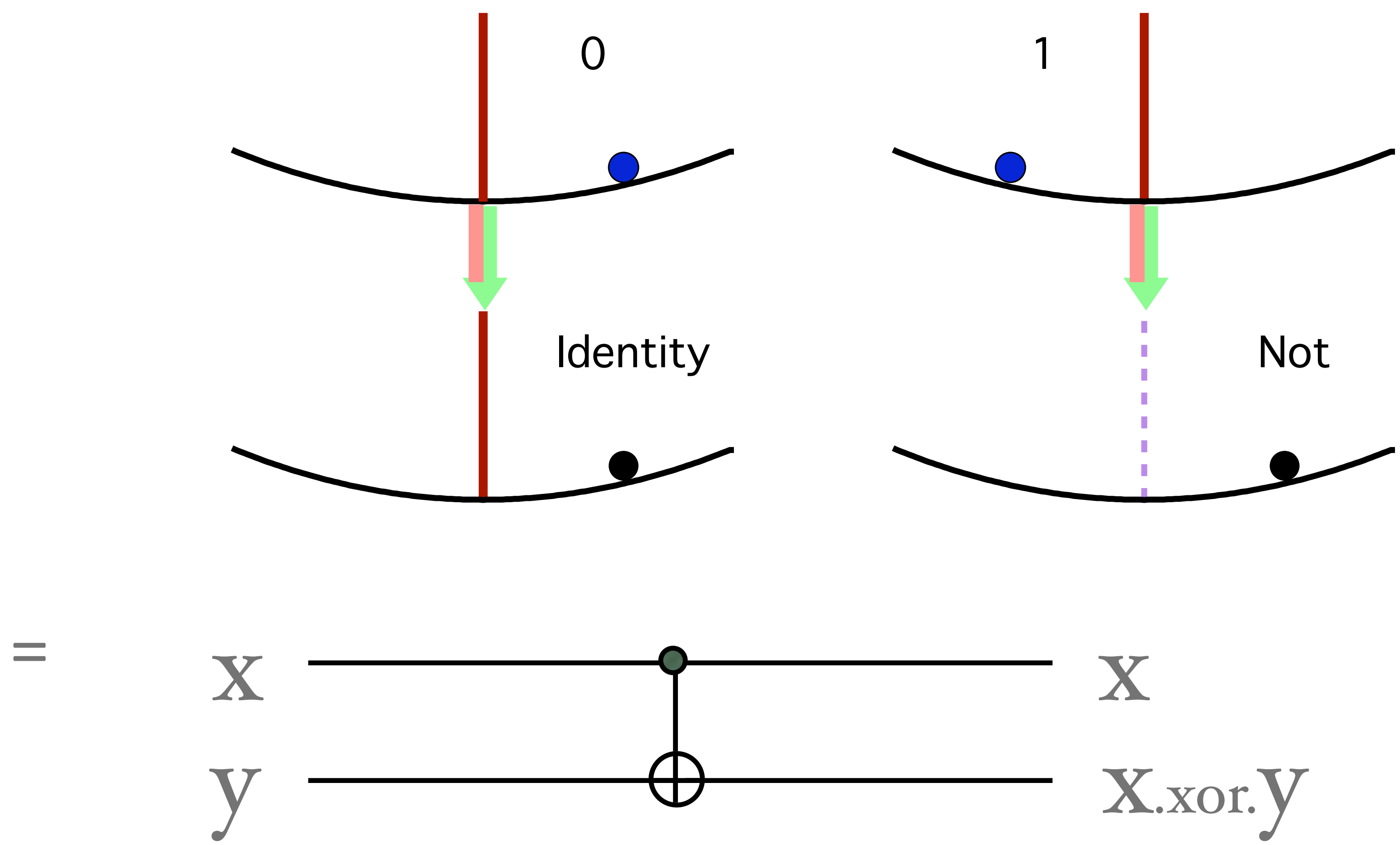
- opening/closing of hard wall GATE as logic operation in time-step $\tau = \pi/\omega$

half oscillatior period



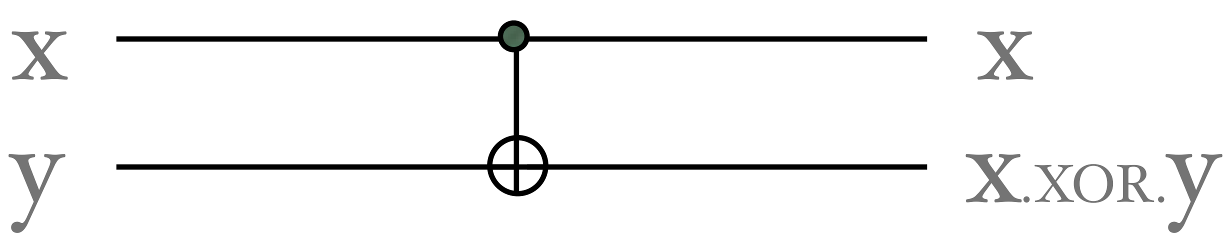
Two-bit trigger

- A two-bit trigger for C-Not operation

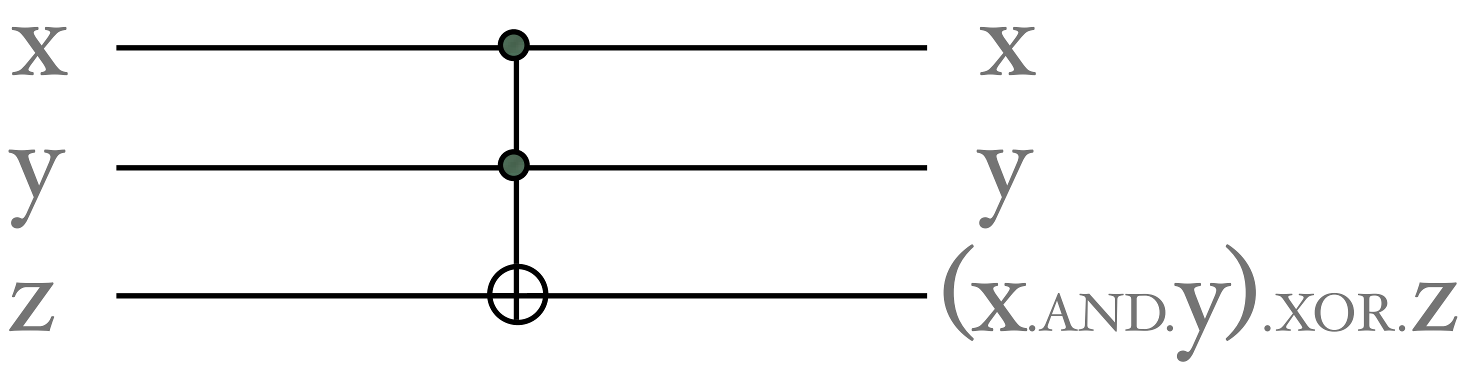


Three-bit trigger

- In a similar manner to CNTL-Not gate



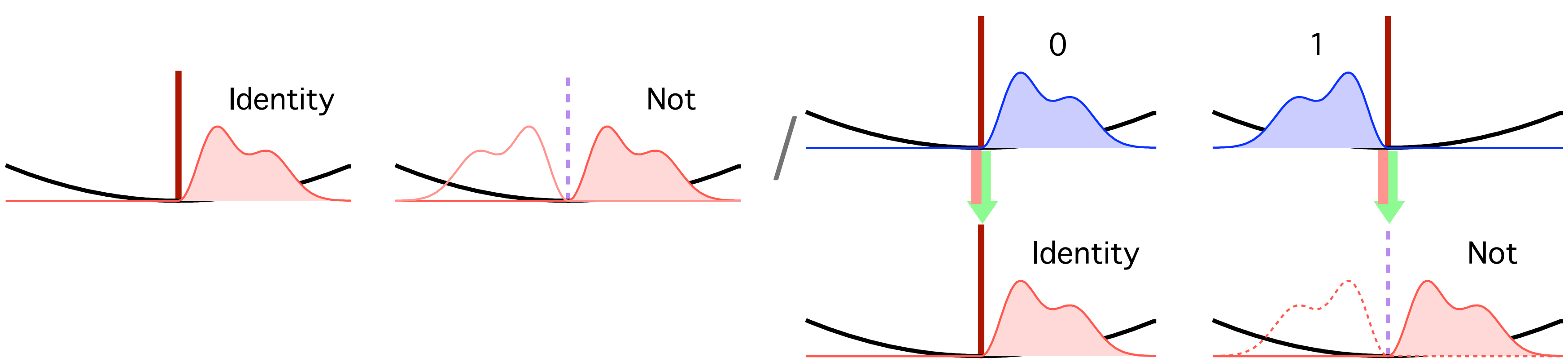
three-bit trigger gate can be designed to creat universal CNTL-CNTL-Not gate



Quantum waves

- works just fine if localized in one side

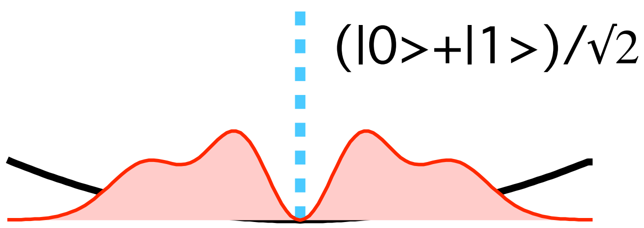
… with same gate and trigger



- … since states get mirrored after τ:
  $\psi(x,\tau) = \pm\, \psi(x,0)$ with h.o. potential

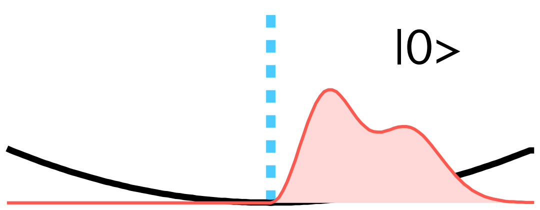
Qubit superposition

- In general, however, a quantum state is bilocalized

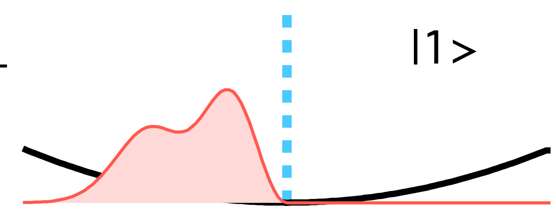


describable as sum of QUBITS $|0\rangle$ and $|1\rangle$

$= \sqrt{\frac{1}{2}}$



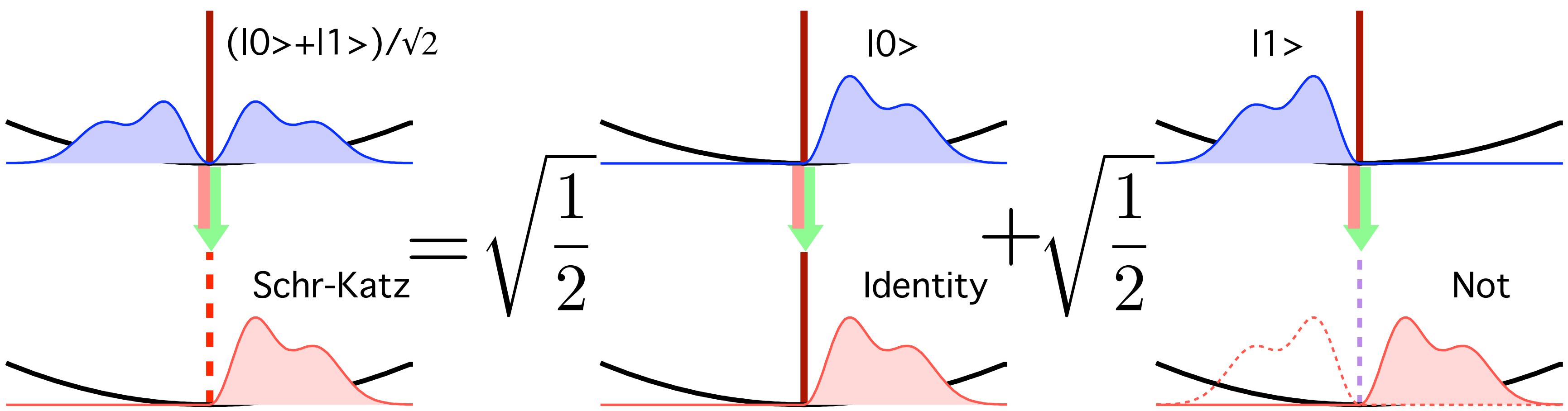
$+ \sqrt{\frac{1}{2}}$



- An advantage might be taken

Schrödinger-cat gate

- If, instead, bilocal wave is used for trigger gate



...is conditionally half open/closed gate

- parallel multiple operation

U(2) Hilbert space

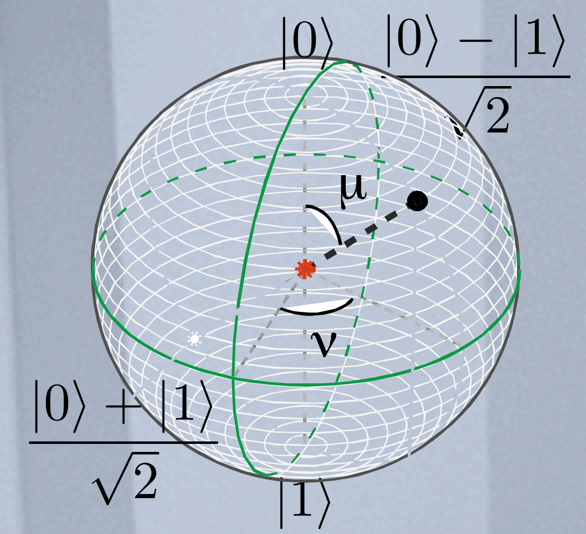
- Quantum information processing possible, if we realize/control U(2) states

$$|\Psi\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$$

- Essentially, apart from overall phase, need to realize Bloch sphere $\{\mu, \nu\}$

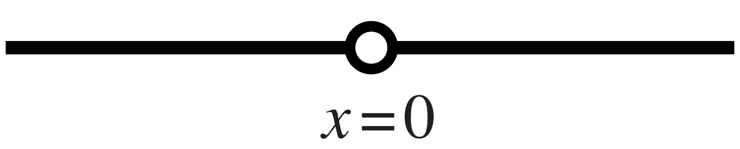
$$|\Psi\rangle = \cos\frac{\mu}{2}\,|0\rangle + e^{i\nu}\sin\frac{\mu}{2}\,|1\rangle$$

- This is achieved by ......



Quantum barriers

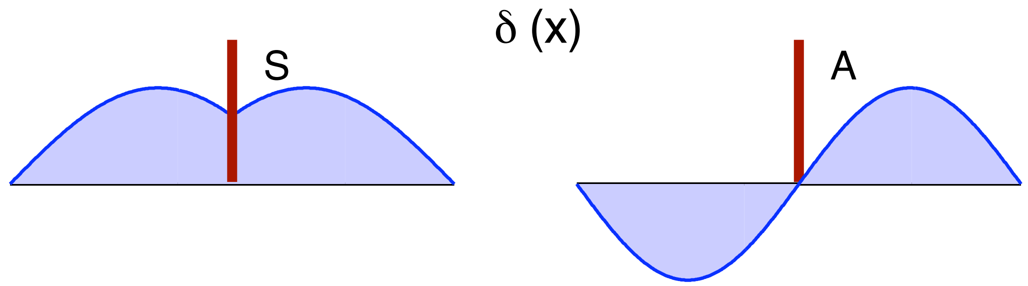
- **Regulate the temporal developement** of wave function at each time-step τ by controlling the **characteristics** of ...
- Quantum mechanically possible **thin barriers**, which is idealized as ...
- 1D **Quantum Point Interactions** ...



- that generalize open/closed gate

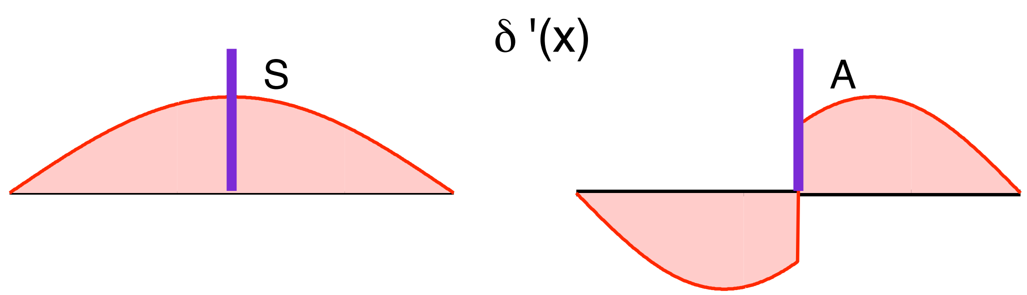
Some point interactions

- Delta-interaction : 1-parameter family



Discontinuity in $\psi'(x)$ » Dirichlet wall

- Delta’-interaction : 1-parameter family



Discontinuity in $\psi(x)$ » Neumann wall

Matrix notations

- Express qubits as 2-vectors

$$|0\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix} \qquad |1\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}$$

- Some Bloch-sphere elements

$$I = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ : Identity} \qquad \sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \text{ : Not}$$

$$h = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \text{ : Hadamard}$$

U(2) point interactions

- Right/left vectors

$$\Psi(x) = \begin{pmatrix} \psi(x) \\ \psi(-x) \end{pmatrix}, \Psi'(x) = \begin{pmatrix} \psi'(x) \\ -\psi'(-x) \end{pmatrix}, \quad (x > 0)$$

- Current conservation at x=0 gives the connection condition

$$(U - I)\Psi(0) + iL_0(U + I)\Psi'(0) = 0$$

- $U \in U(2) \approx S^1 \times S^3$: 4-parameter family

arbitrary scale parameter

SU(2) algebra

- Consider transformations on $\psi(x)$
  P_1: $\psi(x) \to \psi(-x)$: mirror-reflection
  P_2: $\psi(x) \to -i(\Theta(x)-\Theta(-x))\psi(-x)$
  P_3: $\psi(x) \to (\Theta(x)-\Theta(-x))\psi(x)$: half-flip
- They act on Ψ as Pauli matrices
  P_1: $\Psi(x) \to \sigma_1 \Psi(x)$
  P_2: $\Psi(x) \to \sigma_2 \Psi(x)$
  P_3: $\Psi(x) \to \sigma_3 \Psi(x)$

Spectral invariance

- Let H_U be Hamiltonian with U and let σ be one of σi
- $U' = \sigma\, U\, \sigma$ <--> $H_{U'} = P\, H_U P$
- H_U and $H_{U'}$ share the same spectra because $H_{U'} P = P\, H_U$

  $H_U \psi = E\psi$ <--> $H_{U'}(P\psi) = E\,(P\psi)$
- Spectra invariant for all "spin rotation"

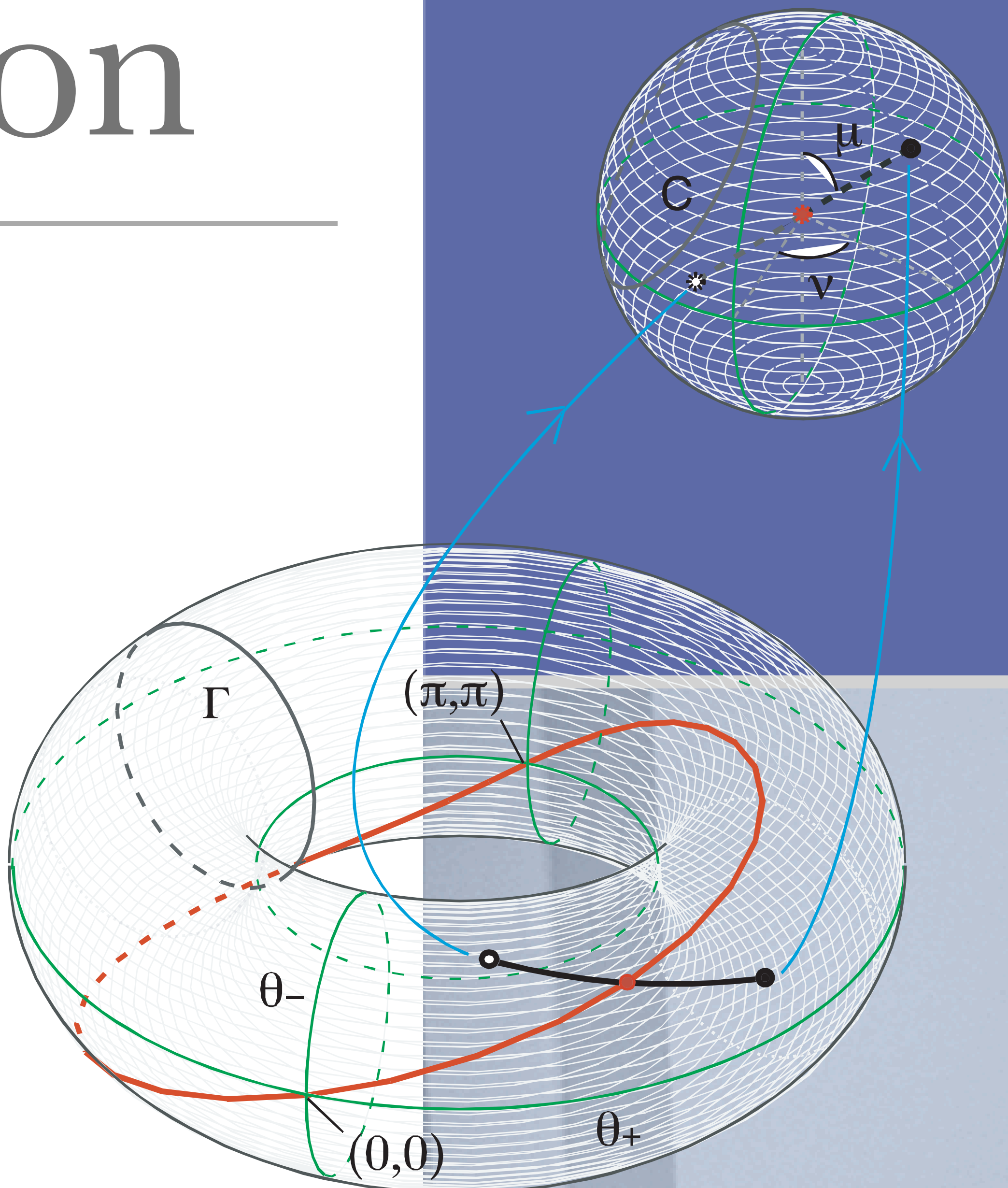
Spectral Decomposition

- Any element U is expressed as

$$U = \Sigma D \Sigma$$

$$\Sigma = \begin{pmatrix} \cos\frac{\mu}{2} & e^{i\nu}\sin\frac{\mu}{2} \\ e^{-i\nu}\sin\frac{\mu}{2} & -\cos\frac{\mu}{2} \end{pmatrix}$$

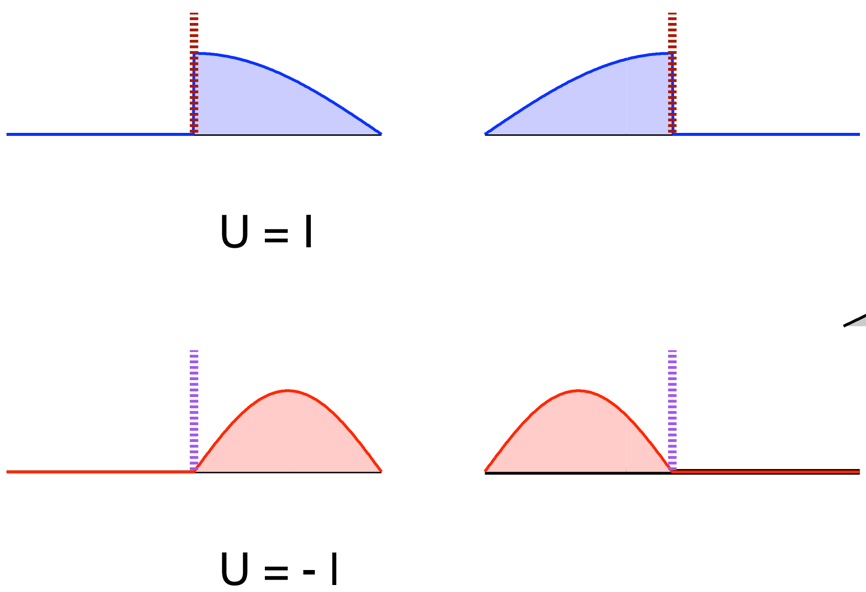
$$D = \begin{pmatrix} e^{i\theta_+} & 0 \\ 0 & e^{i\theta_-} \end{pmatrix}$$

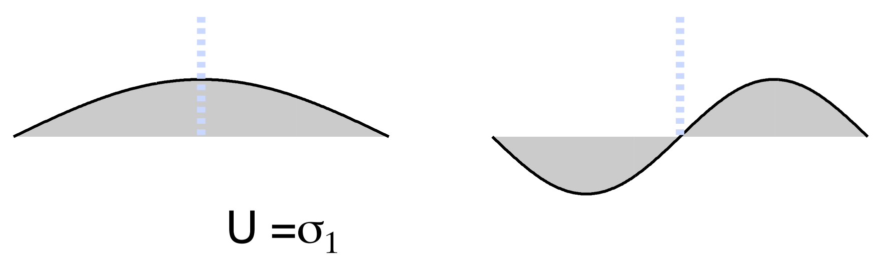


$$\{\mu, \nu\} \in S^2 \ : \text{isospectral}\,, \ \{\theta_+, \theta_-\} \in T^2 \ : \text{spectral}$$

Quantum hard-barriers

- Neumann and Dirichlet walls corresponds to $U = I$ and $U = -I$





- No-barrier corresponds to $U = \sigma_1$

Exotic contact forces

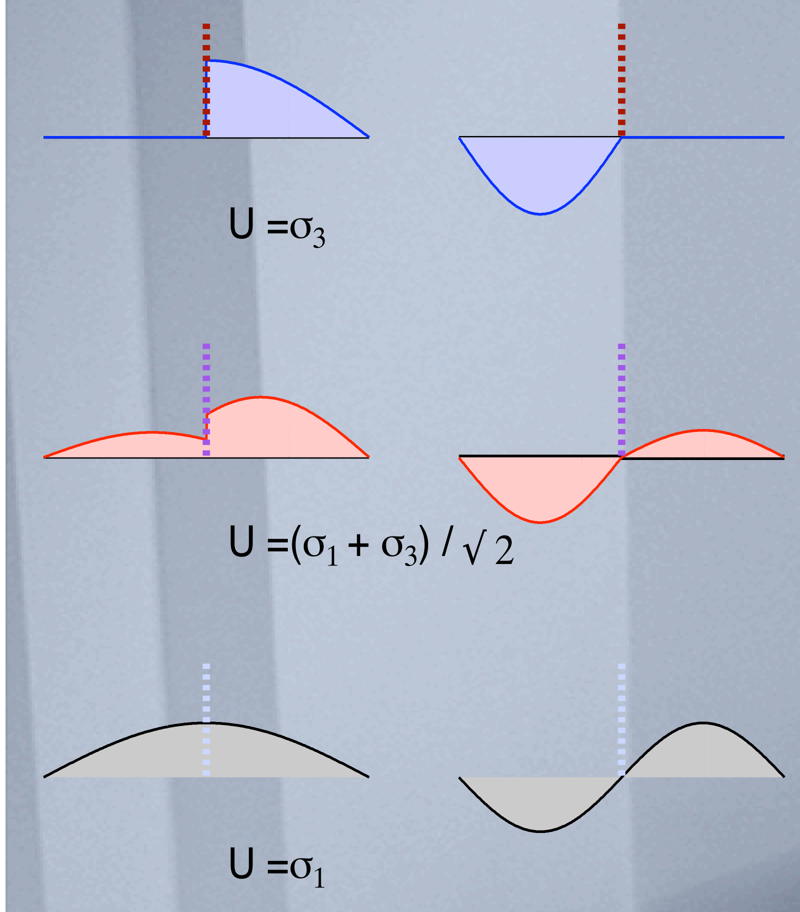
- $U = \sigma_3 \cos\frac{\mu}{2} + \sigma_1 \sin\frac{\mu}{2}\cos\nu + i\sigma_2 \sin\frac{\mu}{2}\sin\nu$

  $U \in S^2(\text{isospectral})$

- Isospectral sphere is “scale-invariant” ››› E-indep. constant transmission

- Interpolate between σ_1 and σ_3

- Treat them as gates to obtain desired mixture of localized wave functions

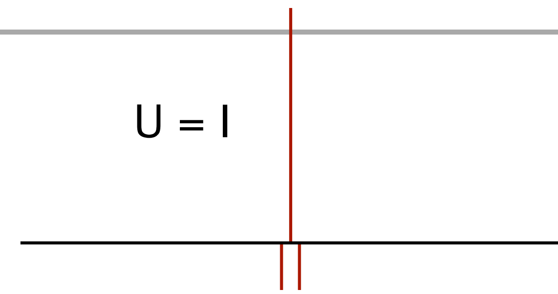


Realizing contact forces

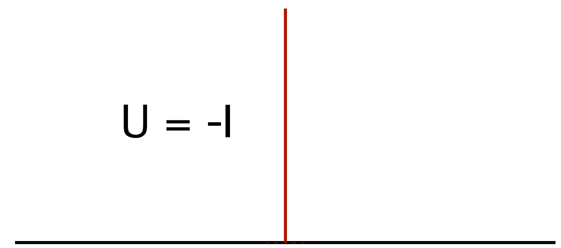
- Delta interaction is a short-range limit of potential with constant volume integral
- How can you produce other exotic point interactions?
- Ans: by singular short-range limit of up to five delta potentials

Examples of realization

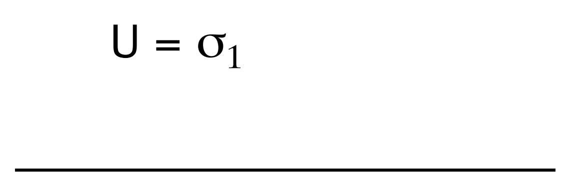
- Exitic delta'



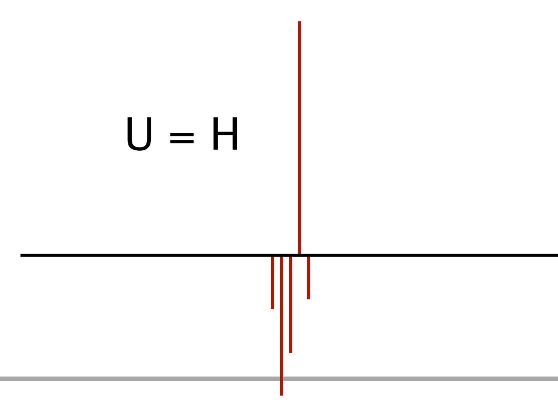
- Dirac delta



- No barrier (free)



- Scale invariant



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Comm. Math. Phys. 224 (2001) 593.

State evolution

- Time evolution of $\Psi(x)$ under given point interaction $U \in S^2$(isospectral)

$$\Psi \underset{t}{\longmapsto} \mathcal{U}_t(U)\,\Psi$$

- With the time step $\tau = \pi/\omega$

  Fact 1 : $\mathcal{U}_\tau(I) = -i$

  Fact 2 : $\mathcal{U}_\tau(\sigma_1) = -i\sigma_1$

  Fact 3 : $\mathcal{U}_\tau(\sigma_3) = -i\sigma_3$

- Parametric S^2 turned into Bloch S^2 ?!

Bloch-sphere imprinting

- Suppose $U' = VUV^{-1}$

  Time evolution under *U* and *U'* are related by $\mathcal{U}_\tau(U') = V\mathcal{U}_\tau(U)V^{-1}$

- $U = V\sigma_3 V^{-1},\ V \in SU(2)$

  span S^2(scale inv)

- Therefore, for $U \in S^2(Bloch)$

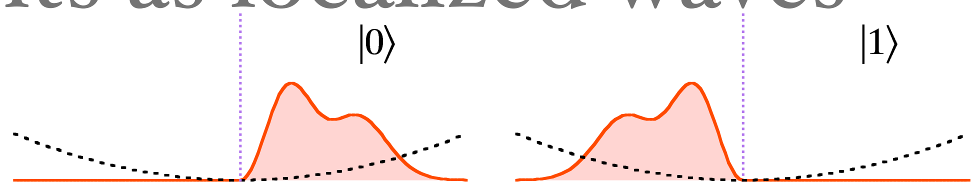
$$\mathcal{U}_\tau(U) = -iU$$

››› Qubit control by quantum gate

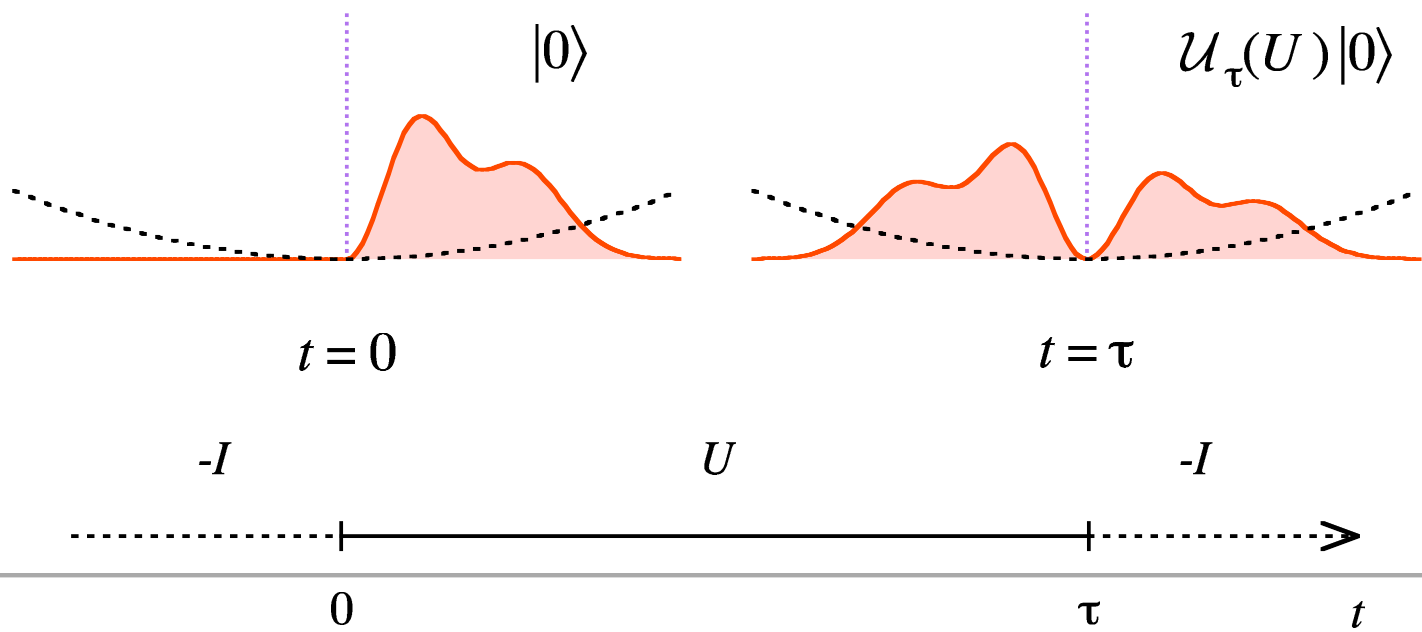
Quantum abacus

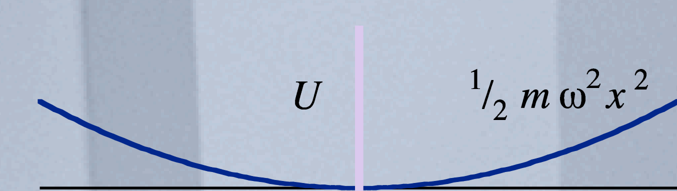
- Harmonic oscillator and *U*

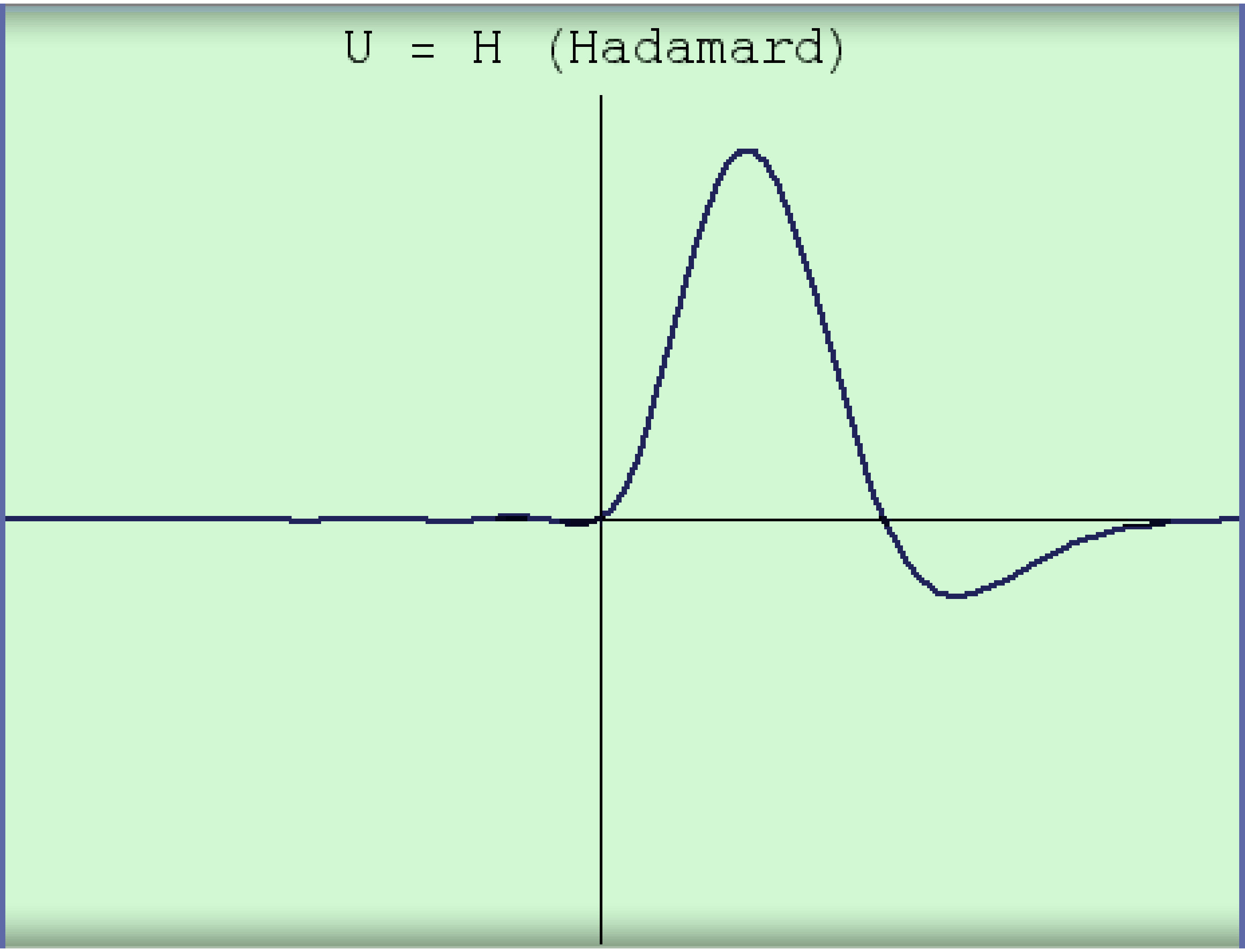
  Qubits as localized waves



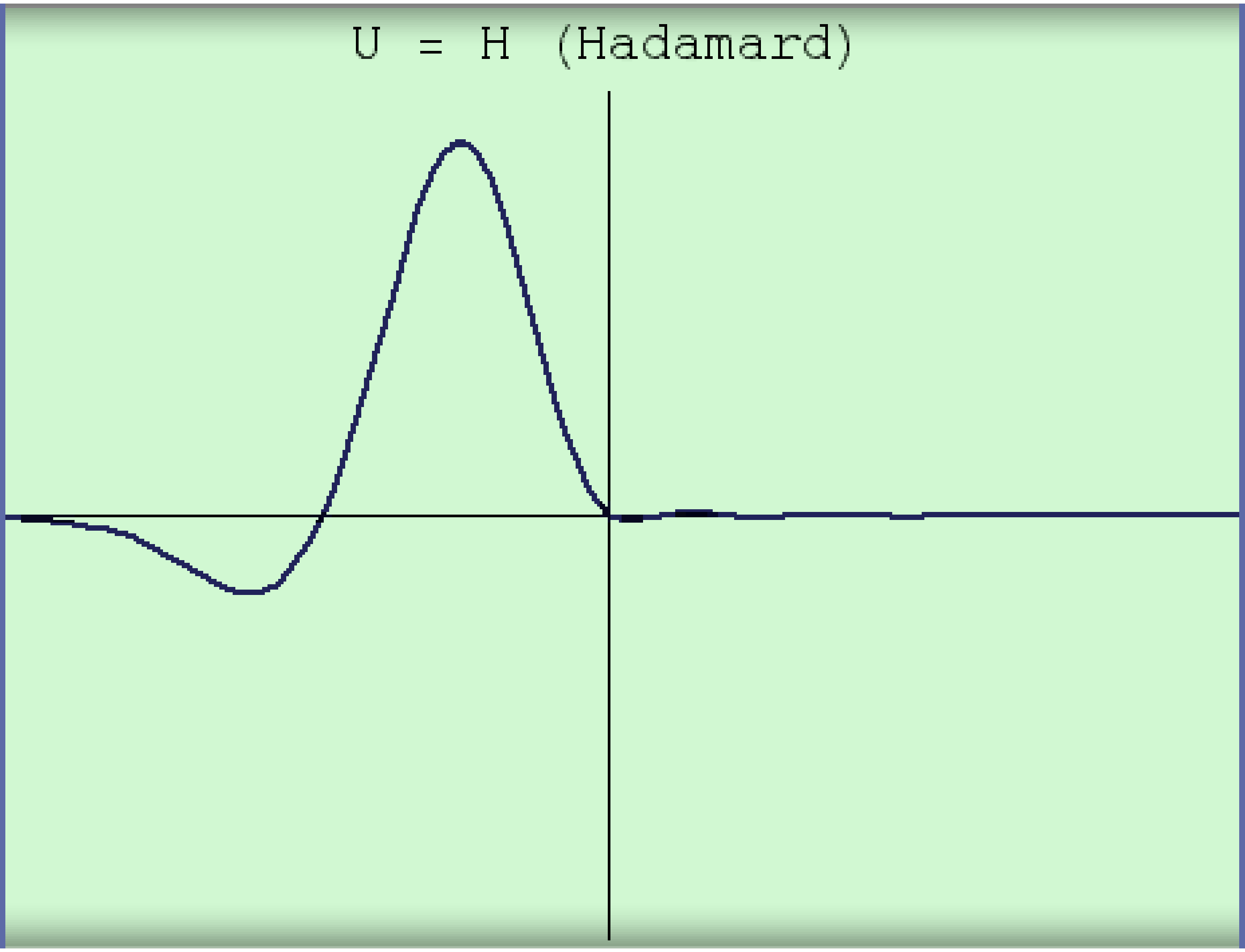
- Evolve under *U* for period *t* to get *-iU*



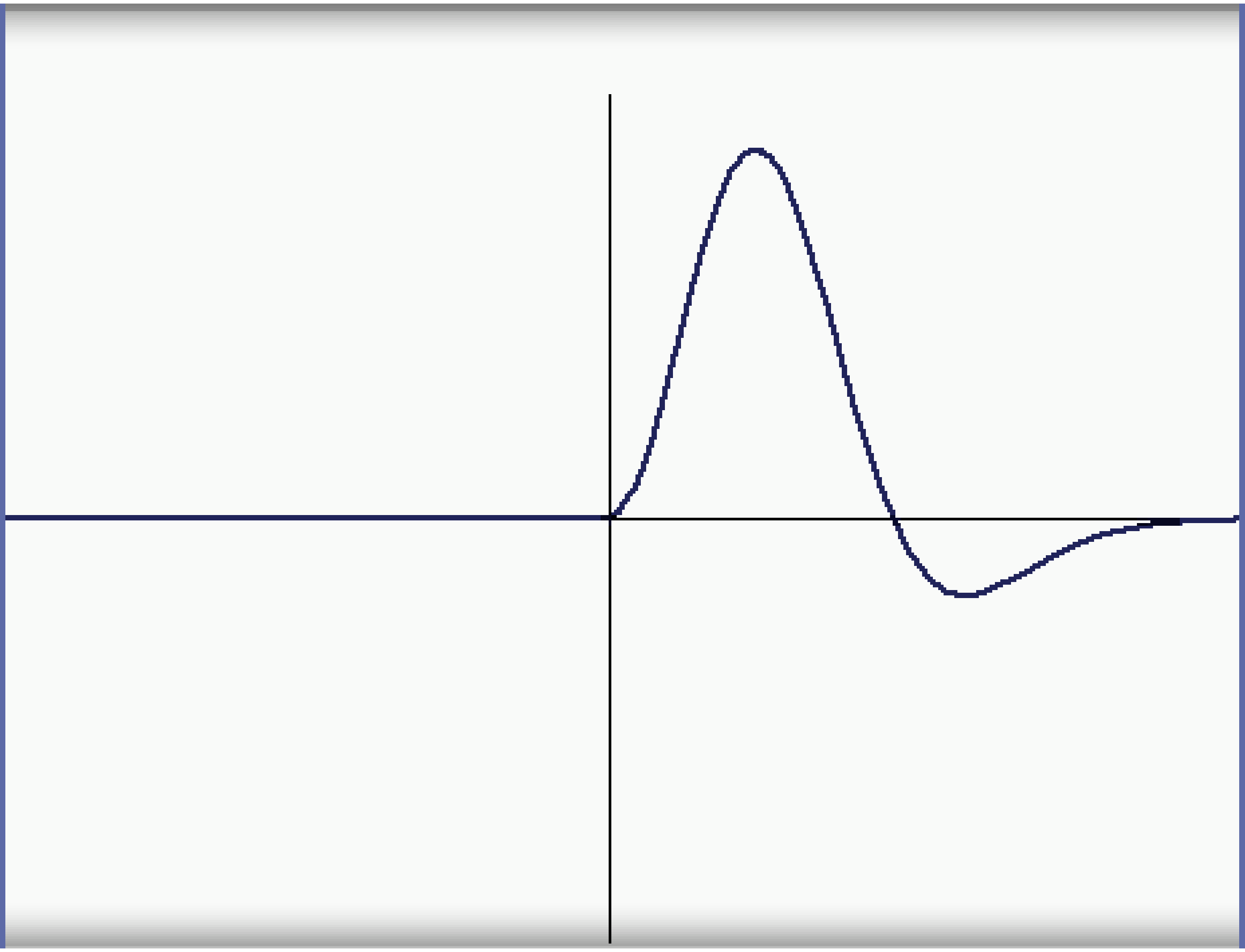




Movie: Quantum Operations U|0>



Movie:
Quantum Operations U|1>



Movie:
Abacus Operation ..$U_3U_2U_1|0>$

Summary

- Quantum qubit is constructed on 1D line with harmonic potential which is divided by a barrier, or ...
- Quantum point interaction, whose mathematical structure is U(2).
- Parameter space S^2 of scale invariant barriers is turned into Hilbert space sphere through Bloch imprinting

References

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