

# Probabilistic analysis of three-player symmetric quantum games played using the Einstein-Podolsky-Rosen-Bohm setting

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## Abstract

This paper extends our probabilistic framework for two-player quantum games to the multiplayer case, while giving a unified perspective for both classical and quantum games. Considering joint probabilities in the standard Einstein-Podolsky-Rosen-Bohm (EPR-Bohm) setting for three observers, we use this setting in order to play general three-player non-cooperative symmetric games. We analyze how the peculiar non-factorizable joint probabilities provided by the EPR-Bohm setting can change the outcome of a game, while requiring that the quantum game attains a classical interpretation for factorizable joint probabilities. In this framework, our analysis of the three-player generalized Prisoner's Dilemma (PD) shows that the players can indeed escape from the classical outcome of the game, because of non-factorizable joint probabilities that the EPR setting can provide. This surprising result for three-player PD contrasts strikingly with our earlier result for two-player PD, played in the same framework, in which even non-factorizable joint probabilities do not result in escaping from the classical consequence of the game.

# 1 Introduction

Although multiplayer games have been extensively studied in the game theory literature [1,2] their analysis is often found more complex than for two-player games [3]. Economics [4] and mathematical biology [5–7] are the areas where most applications of multiplayer games are discussed.

Quantum games first came into prominence following the work of Meyer [16] and Eisert et al [17]. However, the first game-like situations involving many agents were brought to the quantum regime in 1990 by Mermin [8, 9]. Mermin analyzed a multiplayer game that can be won with certainty when it is played using spin-half particles in a Greenberger-Horne-Zeilinger (GHZ) state [10], while no classical strategy can achieve this.

In 1999, Vaidman [11, 12] described the GHZ paradox [10] using a game among three players. Vaidman’s game is now well known in the quantum game literature. In a similar vein, others [13, 14] have discussed game-like situations involving many players for which quantum mechanics significantly helps their chances of winning.

The motivation behind these developments [15] was to use a game framework in order to demonstrate the remarkable, and often counterintuitive, quantum correlations that may arise when many agents interact while sharing quantum resources.

Systematic procedures were suggested [18–34] to quantize a given game and earlier work considered noncooperative games in their normal-form [1].

The approach towards playing quantum games is distinct in that, instead of inventing games in which quantum correlations help players winning games, it proposes quantization procedures for given and very often well known games. That is, instead of tailoring ‘winning conditions’ for the invented games, which can be satisfied when the game is played by quantum players, quantum game theory finds how the sharing of quantum resources may replace/displace/change the solution(s) or the outcome(s) of known games. The emphasis is, therefore, shifted from inventing games to the writing of prescription(s) for quantizing well-known games.

In the area of quantum games, multiplayer games were first studied by Benjamin and Hayden [18] and have subsequently been considered by many others [35–42].

This paper extends to multiplayer quantum games the probabilistic framework originally proposed for two-player noncooperative games by present authors in Ref. [43]. This framework unifies classical and quantum two-player games while using the Einstein-Podolsky-Rosen-Bohm (EPR-Bohm) experiments [44–51] to play a two-player game.

It has been reported in literature [52–54] that joint probabilities in EPR-Bohm experiments may not be factorizable. This led us in Ref. [43] to construct quantum games from non-factorizable joint probabilities. To ensure that the classical game remains embedded within the corresponding quantum game, this framework requires that quantum game attains classical interpretation when joint probabilities are factorizable.

As this framework proposes an entirely probabilistic argument in the construction of quantum games it thus provides a unifying perspective for both classical and quantum games. It also presents a more accessible analysis of quantum games, which can be of potential interest to readers outside the quantum physics domain.

## 2 Three-player symmetric games

We consider three-player symmetric (noncooperative) games in which three players (henceforth labelled as Alice, Bob, and Chris) make their choices simultaneously. The players are assumed located at distance and that they are unable to communicate among themselves. They, however, can communicate to a referee who organizes the game and ensures that the rules of the game are obeyed by the participating parties.

We assume that each player has two choices that we refer to as his/her pure strategies. The payoff relations are made public by the referee at the start of the game. At some particular instant the players are asked to inform the referee of their strategies. The players’ payoffs depend on the

game they play, their strategies, and the set of joint probabilities relevant to the probabilistic system which they share in order to play the game.

We assume Alice's strategies are  $S_1, S_2$ ; Bob's strategies are  $S'_1, S'_2$ ; and Chris' strategies are  $S''_1, S''_2$ , and that we have the following pure-strategy payoff relations [38] that define the game

$$\begin{aligned}
\Pi_{A,B,C}(S_1, S'_1, S''_1) &= \alpha_1, \beta_1, \gamma_1; & \Pi_{A,B,C}(S_1, S'_2, S''_2) &= \alpha_5, \beta_5, \gamma_5; \\
\Pi_{A,B,C}(S_2, S'_1, S''_1) &= \alpha_2, \beta_2, \gamma_2; & \Pi_{A,B,C}(S_2, S'_1, S''_2) &= \alpha_6, \beta_6, \gamma_6; \\
\Pi_{A,B,C}(S_1, S'_2, S''_1) &= \alpha_3, \beta_3, \gamma_3; & \Pi_{A,B,C}(S_2, S'_2, S''_1) &= \alpha_7, \beta_7, \gamma_7; \\
\Pi_{A,B,C}(S_1, S'_1, S''_2) &= \alpha_4, \beta_4, \gamma_4; & \Pi_{A,B,C}(S_2, S'_2, S''_2) &= \alpha_8, \beta_8, \gamma_8,
\end{aligned} \tag{1}$$

where the subscripts  $A, B$ , and  $C$  refer to Alice, Bob, and Chris, respectively.

In one of these payoff relations, the three entries in brackets on left side are Alice's, Bob's, and Chris' pure strategies, respectively, and the three numbers on the right side are their rewards, respectively. For example,  $\Pi_{A,B,C}(S_1, S'_2, S''_1) = \alpha_3, \beta_3, \gamma_3$  states that Alice, Bob, and Chris obtain  $\alpha_3, \beta_3, \gamma_3$ , respectively, when they play the strategies  $S_1, S'_2, S''_1$ , respectively.

In repeated runs, a player can choose between his/her two pure strategies with some probability, which defines his/her mixed-strategy. We denote a mixed-strategy by  $x, y, z \in [0, 1]$  for Alice, Bob, and Chris, respectively. These are probabilities with which Alice, Bob, and Chris play the pure strategies  $S_1, S'_1, S''_1$ , respectively. They, then, play the pure strategies  $S_2, S'_2, S''_2$  with probabilities  $(1-x), (1-y), (1-z)$ , respectively, and the mixed-strategy payoff relations, therefore, read

$$\begin{aligned}
\Pi_{A,B,C}(x, y, z) &= xyz\Pi_{A,B,C}(S_1, S'_1, S''_1) + x(1-y)z\Pi_{A,B,C}(S_1, S'_2, S''_1) + \\
&xy(1-z)\Pi_{A,B,C}(S_1, S'_1, S''_2) + x(1-y)(1-z)\Pi_{A,B,C}(S_1, S'_2, S''_2) + \\
&(1-x)yz\Pi_{A,B,C}(S_2, S'_1, S''_1) + (1-x)(1-y)z\Pi_{A,B,C}(S_2, S'_2, S''_1) + \\
&(1-x)y(1-z)\Pi_{A,B,C}(S_2, S'_1, S''_2) + (1-x)(1-y)(1-z)\Pi_{A,B,C}(S_2, S'_2, S''_2).
\end{aligned} \tag{2}$$

In this paper we consider symmetric three-player games that are defined by the condition that a player's payoff is decided by his/her strategy and not by his/her identity. Mathematically, this is expressed by the conditions

$$\Pi_A(x, y, z) = \Pi_A(x, z, y) = \Pi_B(y, x, z) = \Pi_B(z, x, y) = \Pi_C(y, z, x) = \Pi_C(z, y, x). \tag{3}$$

The payoff relations (2) satisfy these conditions (3) when [38],

$$\begin{aligned}
\beta_1 &= \alpha_1, & \beta_2 &= \alpha_3, & \beta_3 &= \alpha_2, & \beta_4 &= \alpha_3, \\
\beta_5 &= \alpha_6, & \beta_6 &= \alpha_5, & \beta_7 &= \alpha_6, & \beta_8 &= \alpha_8, \\
\gamma_1 &= \alpha_1, & \gamma_2 &= \alpha_3, & \gamma_3 &= \alpha_3, & \gamma_4 &= \alpha_2, \\
\gamma_5 &= \alpha_6, & \gamma_6 &= \alpha_6, & \gamma_7 &= \alpha_5, & \gamma_8 &= \alpha_8, \\
\alpha_6 &= \alpha_7, & \alpha_3 &= \alpha_4.
\end{aligned} \tag{4}$$

A symmetric three-player game can, therefore, be defined by only six constants  $\alpha_1, \alpha_2, \alpha_3, \alpha_5, \alpha_6$ , and  $\alpha_8$ . In the rest of this paper we will define these six constants to be  $\alpha, \beta, \delta, \epsilon, \theta, \omega$  where  $\alpha_1 = \alpha, \alpha_2 = \beta, \alpha_3 = \delta, \alpha_5 = \epsilon, \alpha_6 = \theta$ , and  $\alpha_8 = \omega$ . The pure-strategy payoff relations (1) are then re-expressed as

$$\begin{aligned}
\Pi_{A,B,C}(S_1, S'_1, S''_1) &= \alpha, \alpha, \alpha; & \Pi_{A,B,C}(S_1, S'_2, S''_2) &= \epsilon, \theta, \theta; \\
\Pi_{A,B,C}(S_2, S'_1, S''_1) &= \beta, \delta, \delta; & \Pi_{A,B,C}(S_2, S'_1, S''_2) &= \theta, \epsilon, \theta; \\
\Pi_{A,B,C}(S_1, S'_2, S''_1) &= \delta, \beta, \delta; & \Pi_{A,B,C}(S_2, S'_2, S''_1) &= \theta, \theta, \epsilon; \\
\Pi_{A,B,C}(S_1, S'_1, S''_2) &= \delta, \delta, \beta; & \Pi_{A,B,C}(S_2, S'_2, S''_2) &= \omega, \omega, \omega.
\end{aligned} \tag{5}$$

## 2.1 Three-player Prisoner's Dilemma

Prisoner's Dilemma (PD) is a noncooperative game [1] that is widely known in the areas of economics, social, and political sciences. In recent years quantum physics has been added to this list. This game was investigated [17] early in the history of quantum games and is known to have had provided significant motivation for further work in this area.

Two-player PD is about two criminals (referred hereafter as players) who are arrested after having committed a crime. The investigators have the following plan to make them confess their crime. Both are placed in separate cells and are not allowed to communicate. They are contacted individually and are asked to choose between two choices (strategies): *to confess* ( $D$ ) and *not to confess* ( $C$ ), where  $C$  and  $D$  stand for Cooperation and Defection and this well-known wording of their available choices refers to the fellow prisoner and not to the authorities.

The rules state that if neither prisoner confesses, i.e.  $(C, C)$ , both are given freedom; when one prisoner confesses ( $D$ ) and the other does not ( $C$ ), i.e.  $(C, D)$  or  $(D, C)$ , the prisoner who confesses gets freedom as well as financial reward, while the prisoner who did not confess ends up in the prison for a longer term. When both prisoners confess, i.e.  $(D, D)$ , both are given a reduced term.

In the two-player case the strategy pair  $(D, D)$  comes out as the unique NE (and the rational outcome) of the game, leading to the situation of both having reduced term. The game offers a dilemma as the rational outcome  $(D, D)$  differs from the outcome  $(C, C)$ , which is an available choice, and for which both prisoners get freedom.

The three-player PD is defined by making the association:

$$S_1 \sim C, S_2 \sim D, S'_1 \sim C, S'_2 \sim D, S''_1 \sim C, S''_2 \sim D \quad (6)$$

and afterwards imposing the following conditions [55]:

a) The strategy  $S_2$  is a dominant choice [1] for each player. For Alice this requires:

$$\begin{aligned} \Pi_A(S_2, S'_1, S''_1) &> \Pi_A(S_1, S'_1, S''_1), \\ \Pi_A(S_2, S'_2, S''_2) &> \Pi_A(S_1, S'_2, S''_2), \\ \Pi_A(S_2, S'_1, S''_2) &> \Pi_A(S_1, S'_1, S''_2), \end{aligned} \quad (7)$$

and similar inequalities hold for players Bob and Chris.

b) A player is better off if more of his opponents choose to cooperate. For Alice this requires:

$$\begin{aligned} \Pi_A(S_2, S'_1, S''_1) &> \Pi_A(S_2, S'_1, S''_2) > \Pi_A(S_2, S'_2, S''_2), \\ \Pi_A(S_1, S'_1, S''_1) &> \Pi_A(S_1, S'_1, S''_2) > \Pi_A(S_1, S'_2, S''_2), \end{aligned} \quad (8)$$

and similar relations hold for Bob and Chris.

c) If one player's choice is fixed, the other two players are left in the situation of a two-player PD. For Alice this requires:

$$\begin{aligned} \Pi_A(S_1, S'_1, S''_2) &> \Pi_A(S_2, S'_2, S''_2), \\ \Pi_A(S_1, S'_1, S''_1) &> \Pi_A(S_2, S'_1, S''_2), \\ \Pi_A(S_1, S'_1, S''_2) &> (1/2) \{ \Pi_A(S_1, S'_2, S''_2) + \Pi_A(S_2, S'_1, S''_2) \}, \\ \Pi_A(S_1, S'_1, S''_1) &> (1/2) \{ \Pi_A(S_1, S'_1, S''_2) + \Pi_A(S_2, S'_1, S''_1) \}, \end{aligned} \quad (9)$$

and similar relations hold for Bob and Chris.

Translating the above conditions while using the notation introduced in (5) requires:

$$\begin{aligned} \text{a) } &\beta > \alpha, \quad \omega > \epsilon, \quad \theta > \delta \\ \text{b) } &\beta > \theta > \omega, \quad \alpha > \delta > \epsilon \\ \text{c) } &\delta > \omega, \quad \alpha > \theta, \quad \delta > (1/2)(\epsilon + \theta), \quad \alpha > (1/2)(\delta + \beta) \end{aligned} \quad (10)$$

which define a generalized three-player PD. For example [55], by letting  $\alpha = 7$ ,  $\beta = 9$ ,  $\delta = 3$ ,  $\epsilon = 0$ ,  $\omega = 1$ ,  $\theta = 5$  all of these conditions hold.

### 3 Playing three-player games using coins

We consider the situation when three players share a probabilistic system to play the three-player symmetric game defined in the Section (2). For this system, in a run, a player has to choose between one out of two pure strategies and, in either case, the outcome (of some measurement, or observation, which follows after players have made their choices) is either  $+1$  or  $-1$ .

When we associate +1 with the Head and -1 with the Tail of a coin, sharing coins (not necessarily unbiased) provides a physical realization of a probabilistic physical system. In the following we consider two setups, both of which use coins in order to play the symmetric three-player game (5). We find that the later setup provides an appropriate arrangement for introducing joint probabilities (associated with an EPR-Bohm setting involving three observers) in the playing of a three-player game.

We note that in the standard EPR-Bohm setting, in a run, each one of the spatially-separated observers chooses one between two directions. A quantum measurement along the two chosen directions, in a run, generate either +1 or -1 as the outcome. That is, in a run one of the four possible outcomes (+1, +1, +1), (+1, +1, -1), (+1, -1, +1), (+1, -1, -1), (-1, +1, +1), (-1, +1, -1), (-1, -1, +1), (-1, -1, -1) emerges.

### 3.1 Three-coin setup

The most natural scenario for playing a three-player game, when they share a probabilistic physical system that involves three coins, is the one when in a run each player is given a coin in a Head state, and ‘to flip’ or to ‘not to flip’ are the player’s available strategies. We denote Alice’s, Bob’s, and Chris’ strategy ‘to flip’ by  $S_1$ ,  $S'_1$ , and  $S''_1$ , respectively, and likewise, we denote Alice’s, Bob’s, and Chris’ strategy ‘not to flip’ by  $S_2$ ,  $S'_2$ , and  $S''_2$ , respectively. The three coins are then passed to a referee who rewards players after observing the state of the three coins.

In repeated runs, the players Alice, Bob, and Chris can play mixed strategies denoted by  $x, y, z \in [0, 1]$ , respectively. Here  $x, y, z$  are the probabilities to choose  $S_1$  (out of  $S_1$  and  $S_2$ ),  $S'_1$  (out of  $S'_1$  and  $S'_2$ ), and  $S''_1$  (out of  $S''_1$  and  $S''_2$ ), by Alice, Bob, and Chris, respectively:

$$\begin{aligned} \Pi_{A,B,C}(x, y, z) = & xyz(\alpha, \alpha, \alpha) + x(1-y)z(\delta, \beta, \delta) + xy(1-z)(\delta, \delta, \beta) + \\ & x(1-y)(1-z)(\epsilon, \theta, \theta) + (1-x)yz(\beta, \delta, \delta) + (1-x)(1-y)z(\theta, \theta, \epsilon) + \\ & (1-x)y(1-z)(\theta, \epsilon, \theta) + (1-x)(1-y)(1-z)(\omega, \omega, \omega). \end{aligned} \quad (11)$$

Assume  $(x^*, y^*, z^*)$  is a Nash Equilibrium (NE) [1] then:

$$\begin{aligned} \Pi_A(x^*, y^*, z^*) - \Pi_A(x, y^*, z^*) &\geq 0, \\ \Pi_B(x^*, y^*, z^*) - \Pi_B(x^*, y, z^*) &\geq 0, \\ \Pi_C(x^*, y^*, z^*) - \Pi_C(x^*, y^*, z) &\geq 0. \end{aligned} \quad (12)$$

In the following, we will use NE when we refer to either a Nash Equilibrium or to Nash Equilibria, as determined by the context. We call this arrangement, which uses three coins to play a three-player game, the *three-coin setup*.

### 3.2 Six-coin setup

The three-player game (5) can also be played using six coins (not necessarily unbiased) instead of the three. This can be arranged as follows. Two coins are assigned to each player before the game begins. In a run each player chooses one out of the two, which defines his/her strategy in the run. Three coins are, therefore, chosen in a run. The three chosen coins are passed to a referee who tosses them together and observes the outcome. Many such outcomes are observed as the process of receiving, choosing, and subsequently tossing the coins is repeated many times.

After many runs, the referee rewards the players according to their strategies (i.e. which coin(s) they have chosen over many runs), the outcomes of several tosses giving rise to the underlying statistics of the coins and from the six coefficients defining the three-player symmetric game defined in Section (2).

Notice that coins are tossed in each run, which gives the playing of a game an inherently probabilistic character. This paves the way to step into the quantum regime and provides the key for introducing quantum probabilities.

We call this arrangement of using six coins, for playing a three-player game, the *six-coin setup*. Why introduce a six-coin setup when a three-player game can also be played in three-coin setup?

The answer is provided by the EPR-Bohm setting that involves three observers and 64 joint probabilities. The six coin setup allows us to translate the playing of a three-player game in terms of 64 joint probabilities. When these joint probabilities are quantum mechanical (and are obtained from an EPR-Bohm setting involving three observers) they might have the unusual character of being non-factorizable. That is, the six-coin setup serves as an intermediate step allowing us to see the impact of non-factorizable quantum probabilities on the solution of a game.

In the six-coin setup, by our definition, a player plays a pure strategy<sup>1</sup> when s/he chooses the same coin over all the runs and s/he plays a mixed strategy when s/he chooses his/her first coin with some probability over the runs.

Notice that, by its construction, this setup requires a large number of runs for playing a game, irrespective of whether players play the pure strategies or the mixed strategies, as in either case players' payoffs depend on outcomes of many tosses.

We denote Alice's two coins by  $S_1, S_2$ ; Bob's two coins by  $S'_1, S'_2$ ; and Chris' two coins by  $S''_1, S''_2$ . Heads of a coin is associated (as it is in the three-coin setup) with +1 and tails with -1, and we denote the outcome of Alice's, Bob's, and Chris' coins as  $\pi_A, \pi_B$ , and  $\pi_C$ , respectively.

Alice's outcome of  $\pi_A = +1$  or  $-1$ , whether she goes for the  $S_1$ -coin or the  $S_2$ -coin in a run, is independent of Bob's outcome of  $\pi_B = +1$  or  $-1$  as well as whether he goes for the  $S'_1$ -coin or the  $S'_2$ -coin in the same run. Also, both of these outcomes are independent of Chris' outcome of  $\pi_C = +1$  or  $-1$  as well as whether he goes for the  $S''_1$ -coin or the  $S''_2$ -coin in the same run.

The associated probabilities are, therefore, factorizable in the sense that the probability for a triplet of outcomes can be expressed as the product of probability for each outcome separately. Mathematically, this is expressed by writing joint probabilities as the arithmetic product of their respective marginals, i.e.

$$\Pr(\pi_A, \pi_B, \pi_C; S_{1,2}, S'_{1,2}, S''_{1,2}) = \Pr(\pi_A, S_{1,2}) \Pr(\pi_B, S'_{1,2}) \Pr(\pi_C, S''_{1,2}) \quad (13)$$

where, for example, Bob can set  $S'_{1,2}$  at  $S'_1$  or at  $S'_2$  and the probability  $\Pr(\pi_A, \pi_B, \pi_C; S_2, S'_1, S''_2)$  factorizes to  $\Pr(\pi_A; S_2) \Pr(\pi_B; S'_1) \Pr(\pi_C; S''_2)$ .

As  $S_1, S'_1, S''_1$  are Alice's, Bob's, and Chris' first coins, respectively, we assign probabilities  $r, r', r'' \in [0, 1]$  by defining  $r = \Pr(+1; S_1)$ ,  $r' = \Pr(+1; S'_1)$ , and  $r'' = \Pr(+1; S''_1)$ . Namely,  $r$  is the probability of getting head for (Alice's)  $S_1$ -coin;  $r'$  is the probability of getting head for (Bob's)  $S'_1$ -coin; and  $r''$  is the probability of getting head for (Chris')  $S''_1$ -coin.

Similarly, we assign probabilities  $s, s', s'' \in [0, 1]$  to  $S_2, S'_2, S''_2$  that are Alice's, Bob's, and Chris' second coins, respectively:  $s = \Pr(+1; S_2)$ ,  $s' = \Pr(+1; S'_2)$ ,  $s'' = \Pr(+1; S''_2)$ . Namely,  $s$  is the probability of getting head for (Alice's)  $S_2$ -coin;  $s'$  is the probability of getting head for (Bob's)  $S'_2$ -coin; and  $s''$  is the probability of getting head for (Chris')  $S''_2$ -coin. Factorizability, then, for example, allows us to write  $\Pr(+1, -1, -1; S_2, S'_1, S''_2) = s(1 - r')(1 - s'')$ .

### 3.2.1 Payoff relations and the Nash equilibria

Given how we have defined a 'pure strategy' in the six-coin setup the players' pure-strategy payoff relations can now be written as

$$\begin{aligned} \Pi_{A,B,C}(S_1, S'_1, S''_1) &= (\alpha, \alpha, \alpha)rr'r'' + (\delta, \beta, \delta)r(1 - r')r'' + (\delta, \delta, \beta)rr'(1 - r'') + \\ &(\epsilon, \theta, \theta)r(1 - r')(1 - r'') + (\beta, \delta, \delta)(1 - r)r'r'' + (\theta, \theta, \epsilon)(1 - r)(1 - r')r'' + \\ &(\theta, \epsilon, \theta)(1 - r)r'(1 - r'') + (\omega, \omega, \omega)(1 - r)(1 - r')(1 - r''); \end{aligned} \quad (14)$$

$$\begin{aligned} \Pi_{A,B,C}(S_2, S'_1, S''_1) &= (\alpha, \alpha, \alpha)sr'r'' + (\delta, \beta, \delta)s(1 - r')r'' + (\delta, \delta, \beta)sr'(1 - r'') + \\ &(\epsilon, \theta, \theta)s(1 - r')(1 - r'') + (\beta, \delta, \delta)(1 - s)r'r'' + (\theta, \theta, \epsilon)(1 - s)(1 - r')r'' + \\ &(\theta, \epsilon, \theta)(1 - s)r'(1 - r'') + (\omega, \omega, \omega)(1 - s)(1 - r')(1 - r''); \end{aligned} \quad (15)$$

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<sup>1</sup>This definition of a pure strategy, of course, corresponds to the usual definition of a mixed-strategy, in accordance with the known result in quantum games that a product pure state results in a mixed-strategy classical game.

$$\begin{aligned} \Pi_{A,B,C}(S_1, S'_2, S''_1) &= (\alpha, \alpha, \alpha)rs'r'' + (\delta, \beta, \delta)r(1-s')r'' + (\delta, \delta, \beta)rs'(1-r'') + \\ &(\epsilon, \theta, \theta)r(1-s')(1-r'') + (\beta, \delta, \delta)(1-r)s'r'' + (\theta, \theta, \epsilon)(1-r)(1-s')r'' + \\ &(\theta, \epsilon, \theta)(1-r)s'(1-r'') + (\omega, \omega, \omega)(1-r)(1-s')(1-r''); \end{aligned} \quad (16)$$

$$\begin{aligned} \Pi_{A,B,C}(S_1, S'_1, S''_2) &= (\alpha, \alpha, \alpha)rr's'' + (\delta, \beta, \delta)r(1-r')s'' + (\delta, \delta, \beta)rr'(1-s'') + \\ &(\epsilon, \theta, \theta)r(1-r')(1-s'') + (\beta, \delta, \delta)(1-r)r's'' + (\theta, \theta, \epsilon)(1-r)(1-r')s'' + \\ &(\theta, \epsilon, \theta)(1-r)r'(1-s'') + (\omega, \omega, \omega)(1-r)(1-r')(1-s''); \end{aligned} \quad (17)$$

$$\begin{aligned} \Pi_{A,B,C}(S_1, S'_2, S''_2) &= (\alpha, \alpha, \alpha)rs's'' + (\delta, \beta, \delta)r(1-s')s'' + (\delta, \delta, \beta)rs'(1-s'') + \\ &(\epsilon, \theta, \theta)r(1-s')(1-s'') + (\beta, \delta, \delta)(1-r)s's'' + (\theta, \theta, \epsilon)(1-r)(1-s')s'' + \\ &(\theta, \epsilon, \theta)(1-r)s'(1-s'') + (\omega, \omega, \omega)(1-r)(1-s')(1-s''); \end{aligned} \quad (18)$$

$$\begin{aligned} \Pi_{A,B,C}(S_2, S'_1, S''_2) &= (\alpha, \alpha, \alpha)sr's'' + (\delta, \beta, \delta)s(1-r')s'' + (\delta, \delta, \beta)sr'(1-s'') + \\ &(\epsilon, \theta, \theta)s(1-r')(1-s'') + (\beta, \delta, \delta)(1-s)r's'' + (\theta, \theta, \epsilon)(1-s)(1-r')s'' + \\ &(\theta, \epsilon, \theta)(1-s)r'(1-s'') + (\omega, \omega, \omega)(1-s)(1-r')(1-s''); \end{aligned} \quad (19)$$

$$\begin{aligned} \Pi_{A,B,C}(S_2, S'_2, S''_1) &= (\alpha, \alpha, \alpha)ss'r'' + (\delta, \beta, \delta)s(1-s')r'' + (\delta, \delta, \beta)ss'(1-r'') + \\ &(\epsilon, \theta, \theta)s(1-s')(1-r'') + (\beta, \delta, \delta)(1-s)s'r'' + (\theta, \theta, \epsilon)(1-s)(1-s')r'' + \\ &(\theta, \epsilon, \theta)(1-s)s'(1-r'') + (\omega, \omega, \omega)(1-s)(1-s')(1-r''); \end{aligned} \quad (20)$$

$$\begin{aligned} \Pi_{A,B,C}(S_2, S'_2, S''_2) &= (\alpha, \alpha, \alpha)ss's'' + (\delta, \beta, \delta)s(1-s')s'' + (\delta, \delta, \beta)ss'(1-s'') + \\ &(\epsilon, \theta, \theta)s(1-s')(1-s'') + (\beta, \delta, \delta)(1-s)s's'' + (\theta, \theta, \epsilon)(1-s)(1-s')s'' + \\ &(\theta, \epsilon, \theta)(1-s)s'(1-s'') + (\omega, \omega, \omega)(1-s)(1-s')(1-s''); \end{aligned} \quad (21)$$

where, on right side of each equation, the three constants in brackets correspond to the players Alice, Bob, and Chris respectively. We point out that Eqs. (14-21) are pure-strategy payoff relations, as they represent the situation when each of the three players chooses the same coin for all runs. For example, referring to Eq. (16),  $\Pi_A(S_1, S'_2, S''_1)$  is Alice's payoff when she goes for the  $S_1$ -coin, Bob goes for  $S'_2$ -coin, and Chris goes for  $S''_1$ -coin for all runs.

A mixed-strategy game, in the six-coin setup, corresponds to when, over a large number of runs of the game, a player chooses one of the two available coins with some probability. Let  $x, y,$  and  $z$  be the probabilities with which Alice, Bob, and Chris, respectively, choose the coins  $S_1, S'_1,$  and  $S''_1$ , respectively. The players' six-coin mixed-strategy payoff relations then read,

$$\begin{aligned} \Pi_{A,B,C}(x, y, z) &= xyz\Pi_{A,B,C}(S_1, S'_1, S''_1) + x(1-y)z\Pi_{A,B,C}(S_1, S'_2, S''_1) + \\ &xy(1-z)\Pi_{A,B,C}(S_1, S'_1, S''_2) + x(1-y)(1-z)\Pi_{A,B,C}(S_1, S'_2, S''_2) + \\ &(1-x)yz\Pi_{A,B,C}(S_2, S'_1, S''_1) + (1-x)(1-y)z\Pi_{A,B,C}(S_2, S'_2, S''_1) + \\ &(1-x)y(1-z)\Pi_{A,B,C}(S_2, S'_1, S''_2) + (1-x)(1-y)(1-z)\Pi_{A,B,C}(S_2, S'_2, S''_2). \end{aligned} \quad (22)$$

Notice that the right side of Eq. (22) contains expressions that are given by Eqs. (14-21).

Six-coin mixed-strategy payoff relations (22) are mathematically identical to the three-coin mixed-strategy payoff relations (2). However, these equations are to be interpreted differently as the definitions of what constitutes a strategy in three- and six-coin setups are different. In (2) the numbers  $x, y,$  and  $z$  are the probabilities with which Alice, Bob, and Chris, respectively, flip the coin that s/he receives. Whereas in (22) the numbers  $x, y,$  and  $z$  are the probabilities with which, over repeated runs, Alice, Bob, and Chris choose the  $S_1$ -coin, the  $S'_1$ -coin, and the  $S''_1$ -coin, respectively.

Referring to (22), the triplet  $(x^*, y^*, z^*)$  becomes a NE when the following inequalities hold

$$\begin{aligned} \Pi_A(x^*, y^*, z^*) - \Pi_A(x, y^*, z^*) &\geq 0, \\ \Pi_B(x^*, y^*, z^*) - \Pi_B(x^*, y, z^*) &\geq 0, \\ \Pi_C(x^*, y^*, z^*) - \Pi_C(x^*, y^*, z) &\geq 0, \end{aligned} \quad (23)$$

which, though being mathematically identical to (12), refers to the relations (22) and Eqs. (14-21). Also, these inequalities are to be interpreted in terms of how a strategy is defined in the six-coin setup.

### 3.3 Playing Prisoner's Dilemma

In the following we use the three-coin and the six-coin setups to play PD.

#### 3.3.1 With three-coin setup

In three-coin setup, the inequalities (12) give

$$\begin{aligned} \Pi_A(x^*, y^*, z^*) - \Pi_A(x, y^*, z^*) &\geq 0, \\ \Pi_B(x^*, y^*, z^*) - \Pi_B(x^*, y, z^*) &\geq 0, \\ \Pi_C(x^*, y^*, z^*) - \Pi_C(x^*, y^*, z) &\geq 0, \end{aligned} \quad (24)$$

and  $(x^*, y^*, z^*) = (0, 0, 0)$  comes out as a unique NE at which players' payoffs are  $\Pi_A(0, 0, 0) = \omega$ ,  $\Pi_B(0, 0, 0) = \omega$ , and  $\Pi_C(0, 0, 0) = \omega$ .

#### 3.3.2 With six-coin setup

The NE conditions (23) are evaluated using the payoff relations (22) and the Eqs. (14-21). We consider the case when  $(s, s', s'') = (0, 0, 0)$  i.e. the probabilities of getting head for each player's second coin is zero. This reduces the NE conditions (23) to

$$\begin{aligned} (x^* - x) \{y^* z^* (rr'r'')\Delta_1 + r(z^* r'' + y^* r')\Delta_2 + r\Delta_3\} &\geq 0, \\ (y^* - y) \{x^* z^* (rr'r'')\Delta_1 + r'(z^* r'' + x^* r)\Delta_2 + r'\Delta_3\} &\geq 0, \\ (z^* - z) \{x^* y^* (rr'r'')\Delta_1 + r''(y^* r' + x^* r)\Delta_2 + r''\Delta_3\} &\geq 0, \end{aligned} \quad (25)$$

where  $\Delta_1 = (\alpha - \beta - 2\delta + 2\theta + \epsilon - \omega)$ ,  $\Delta_2 = (\delta - \epsilon - \theta + \omega)$ , and  $\Delta_3 = (\epsilon - \omega)$ . For PD we have  $\Delta_3 < 0$ , it then follows from (25) that, when  $(s, s', s'') = (0, 0, 0)$ , the strategy triplet  $(x^*, y^*, z^*) = (0, 0, 0)$ , as is defined in the six-coin setup, comes out to be a unique NE in the three-player symmetric game of PD. In other words, this is described by saying that the triplet  $(x^*, y^*, z^*) = (0, 0, 0)$  emerges as a unique NE when the probabilities of getting a head for each player's second coin is zero and, along with this, that the joint probabilities are factorizable.

Now the crucial step follows: Requiring coins to satisfy the constraint  $(s, s', s'') = (0, 0, 0)$  can also be translated in terms of constraints on the joint probabilities associated to the six coins. To find these constraints we identify the 64 joint probabilities  $p_1, p_2, \dots, p_{64}$  that can be defined for six coins, and are given as follows:

$$\begin{aligned} p_1 &= rr'r'', & p_5 &= (1-r)r'r'', \\ p_2 &= r(1-r')r'', & p_6 &= (1-r)(1-r')r'', \\ p_3 &= rr'(1-r''), & p_7 &= (1-r)r'(1-r''), \\ p_4 &= r(1-r')(1-r''), & p_8 &= (1-r)(1-r')(1-r''), \end{aligned} \quad (26)$$

$$\begin{aligned} p_9 &= sr'r'', & p_{13} &= (1-s)r'r'', \\ p_{10} &= s(1-r')r'', & p_{14} &= (1-s)(1-r')r'', \\ p_{11} &= sr'(1-r''), & p_{15} &= (1-s)r'(1-r''), \\ p_{12} &= s(1-r')(1-r''), & p_{16} &= (1-s)(1-r')(1-r''), \end{aligned} \quad (27)$$

$$\begin{aligned} p_{17} &= rs'r'', & p_{21} &= (1-r)s'r'', \\ p_{18} &= r(1-s')r'', & p_{22} &= (1-r)(1-s')r'', \\ p_{19} &= rs'(1-r''), & p_{23} &= (1-r)s'(1-r''), \\ p_{20} &= r(1-s')(1-r''), & p_{24} &= (1-r)(1-s')(1-r''), \end{aligned} \quad (28)$$

$$\begin{aligned}
p_{25} &= rr's'', & p_{29} &= (1-r)r's'', \\
p_{26} &= r(1-r')s'', & p_{30} &= (1-r)(1-r')s'', \\
p_{27} &= rr'(1-s''), & p_{31} &= (1-r)r'(1-s''), \\
p_{28} &= r(1-r')(1-s''), & p_{32} &= (1-r)(1-r')(1-s''),
\end{aligned} \tag{29}$$

$$\begin{aligned}
p_{33} &= rs's'', & p_{37} &= (1-r)s's'', \\
p_{34} &= r(1-s')s'', & p_{38} &= (1-r)(1-s')s'', \\
p_{35} &= rs'(1-s''), & p_{39} &= (1-r)s'(1-s''), \\
p_{36} &= r(1-s')(1-s''), & p_{40} &= (1-r)(1-s')(1-s''),
\end{aligned} \tag{30}$$

$$\begin{aligned}
p_{41} &= sr's'', & p_{45} &= (1-s)r's'', \\
p_{42} &= s(1-r')s'', & p_{46} &= (1-s)(1-r')s'', \\
p_{43} &= sr'(1-s''), & p_{47} &= (1-s)r'(1-s''), \\
p_{44} &= s(1-r')(1-s''), & p_{48} &= (1-s)(1-r')(1-s''),
\end{aligned} \tag{31}$$

$$\begin{aligned}
p_{49} &= ss'r'', & p_{53} &= (1-s)s'r'', \\
p_{50} &= s(1-s')r'', & p_{54} &= (1-s)(1-s')r'', \\
p_{51} &= ss'(1-r''), & p_{55} &= (1-s)s'(1-r''), \\
p_{52} &= s(1-s')(1-r''), & p_{56} &= (1-s)(1-s')(1-r''),
\end{aligned} \tag{32}$$

$$\begin{aligned}
p_{57} &= ss's'', & p_{61} &= (1-s)s's'', \\
p_{58} &= s(1-s')s'', & p_{62} &= (1-s)(1-s')s'', \\
p_{59} &= ss'(1-s''), & p_{63} &= (1-s)s'(1-s''), \\
p_{60} &= s(1-s')(1-s''), & p_{64} &= (1-s)(1-s')(1-s'').
\end{aligned} \tag{33}$$

With these definitions the payoff relations (14-21) are re-expressed as

$$\begin{aligned}
\Pi_{A,B,C}(S_1, S'_1, S''_1) &= (\alpha, \alpha, \alpha)p_1 + (\delta, \beta, \delta)p_2 + (\delta, \delta, \beta)p_3 + \\
&+ (\epsilon, \theta, \theta)p_4 + (\beta, \delta, \delta)p_5 + (\theta, \theta, \epsilon)p_6 + (\theta, \epsilon, \theta)p_7 + (\omega, \omega, \omega)p_8;
\end{aligned} \tag{34}$$

$$\begin{aligned}
\Pi_{A,B,C}(S_2, S'_1, S''_1) &= (\alpha, \alpha, \alpha)p_9 + (\delta, \beta, \delta)p_{10} + (\delta, \delta, \beta)p_{11} + \\
&+ (\epsilon, \theta, \theta)p_{12} + (\beta, \delta, \delta)p_{13} + (\theta, \theta, \epsilon)p_{14} + (\theta, \epsilon, \theta)p_{15} + (\omega, \omega, \omega)p_{16};
\end{aligned} \tag{35}$$

$$\begin{aligned}
\Pi_{A,B,C}(S_1, S'_2, S''_1) &= (\alpha, \alpha, \alpha)p_{17} + (\delta, \beta, \delta)p_{18} + (\delta, \delta, \beta)p_{19} + \\
&+ (\epsilon, \theta, \theta)p_{20} + (\beta, \delta, \delta)p_{21} + (\theta, \theta, \epsilon)p_{22} + (\theta, \epsilon, \theta)p_{23} + (\omega, \omega, \omega)p_{24};
\end{aligned} \tag{36}$$

$$\begin{aligned}
\Pi_{A,B,C}(S_1, S'_1, S''_2) &= (\alpha, \alpha, \alpha)p_{25} + (\delta, \beta, \delta)p_{26} + (\delta, \delta, \beta)p_{27} + \\
&+ (\epsilon, \theta, \theta)p_{28} + (\beta, \delta, \delta)p_{29} + (\theta, \theta, \epsilon)p_{30} + (\theta, \epsilon, \theta)p_{31} + (\omega, \omega, \omega)p_{32};
\end{aligned} \tag{37}$$

$$\begin{aligned}
\Pi_{A,B,C}(S_1, S'_2, S''_2) &= (\alpha, \alpha, \alpha)p_{33} + (\delta, \beta, \delta)p_{34} + (\delta, \delta, \beta)p_{35} + \\
&+ (\epsilon, \theta, \theta)p_{36} + (\beta, \delta, \delta)p_{37} + (\theta, \theta, \epsilon)p_{38} + (\theta, \epsilon, \theta)p_{39} + (\omega, \omega, \omega)p_{40};
\end{aligned} \tag{38}$$

$$\begin{aligned}
\Pi_{A,B,C}(S_2, S'_1, S''_2) &= (\alpha, \alpha, \alpha)p_{41} + (\delta, \beta, \delta)p_{42} + (\delta, \delta, \beta)p_{43} + \\
&+ (\epsilon, \theta, \theta)p_{44} + (\beta, \delta, \delta)p_{45} + (\theta, \theta, \epsilon)p_{46} + (\theta, \epsilon, \theta)p_{47} + (\omega, \omega, \omega)p_{48};
\end{aligned} \tag{39}$$

$$\begin{aligned}
\Pi_{A,B,C}(S_2, S'_2, S''_1) &= (\alpha, \alpha, \alpha)p_{49} + (\delta, \beta, \delta)p_{50} + (\delta, \delta, \beta)p_{51} + \\
&+ (\epsilon, \theta, \theta)p_{52} + (\beta, \delta, \delta)p_{53} + (\theta, \theta, \epsilon)p_{54} + (\theta, \epsilon, \theta)p_{55} + (\omega, \omega, \omega)p_{56};
\end{aligned} \tag{40}$$

$$\begin{aligned}
\Pi_{A,B,C}(S_2, S'_2, S''_2) &= (\alpha, \alpha, \alpha)p_{57} + (\delta, \beta, \delta)p_{58} + (\delta, \delta, \beta)p_{59} + \\
&+ (\epsilon, \theta, \theta)p_{60} + (\beta, \delta, \delta)p_{61} + (\theta, \theta, \epsilon)p_{62} + (\theta, \epsilon, \theta)p_{63} + (\omega, \omega, \omega)p_{64}.
\end{aligned} \tag{41}$$

Now, from the definitions (26-33), requiring coins to satisfy the constraint  $(s, s', s'') = (0, 0, 0)$  makes a number of the joint probabilities vanish:

$$\begin{aligned}
p_9 &= 0, p_{10} = 0, p_{11} = 0, p_{12} = 0; \\
p_{17} &= 0, p_{19} = 0, p_{21} = 0, p_{23} = 0; \\
p_{25} &= 0, p_{26} = 0, p_{29} = 0, p_{30} = 0; \\
p_{33} &= 0, p_{34} = 0, p_{35} = 0, p_{37} = 0, p_{38} = 0, p_{39} = 0; \\
p_{41} &= 0, p_{42} = 0, p_{43} = 0, p_{44} = 0, p_{45} = 0, p_{46} = 0; \\
p_{49} &= 0, p_{50} = 0, p_{51} = 0, p_{52} = 0, p_{53} = 0, p_{55} = 0; \\
p_{57} &= 0, p_{58} = 0, p_{59} = 0, p_{60} = 0, p_{61} = 0, p_{62} = 0, p_{63} = 0,
\end{aligned} \tag{42}$$

which, in turn, reduces the pure-strategy payoff relations (34-41) to

$$\begin{aligned}
\Pi_{A,B,C}(S_1, S'_1, S''_1) &= (\alpha, \alpha, \alpha)p_1 + (\delta, \beta, \delta)p_2 + (\delta, \delta, \beta)p_3 + \\
&(\epsilon, \theta, \theta)p_4 + (\beta, \delta, \delta)p_5 + (\theta, \theta, \epsilon)p_6 + (\theta, \epsilon, \theta)p_7 + (\omega, \omega, \omega)p_8; \\
\Pi_{A,B,C}(S_2, S'_1, S''_1) &= (\beta, \delta, \delta)p_{13} + (\theta, \theta, \epsilon)p_{14} + (\theta, \epsilon, \theta)p_{15} + (\omega, \omega, \omega)p_{16}; \\
\Pi_{A,B,C}(S_1, S'_2, S''_1) &= (\delta, \beta, \delta)p_{18} + (\epsilon, \theta, \theta)p_{20} + (\theta, \theta, \epsilon)p_{22} + (\omega, \omega, \omega)p_{24}; \\
\Pi_{A,B,C}(S_1, S'_1, S''_2) &= (\delta, \delta, \beta)p_{27} + (\epsilon, \theta, \theta)p_{28} + (\theta, \epsilon, \theta)p_{31} + (\omega, \omega, \omega)p_{32}; \\
\Pi_{A,B,C}(S_1, S'_2, S''_2) &= (\epsilon, \theta, \theta)p_{36} + (\omega, \omega, \omega)p_{40}; \\
\Pi_{A,B,C}(S_2, S'_1, S''_2) &= (\theta, \epsilon, \theta)p_{47} + (\omega, \omega, \omega)p_{48}; \\
\Pi_{A,B,C}(S_2, S'_2, S''_1) &= (\theta, \theta, \epsilon)p_{54} + (\omega, \omega, \omega)p_{56}; \\
\Pi_{A,B,C}(S_2, S'_2, S''_2) &= (\omega, \omega, \omega)p_{64};
\end{aligned} \tag{43}$$

where only those joint probabilities are left that can have non-zero value(s). These (pure strategy) payoff relations ensure that when the joint probabilities (involved in these expressions) become factorizable the classical outcome of the game results.

## 4 Three-player quantum games

In the quantum game literature [16–23, 27, 28, 30], three-player games have been studied but it appears that they have attracted less attention than the two-player games. This is understandable as their analysis is often found to be significantly harder even in the classical regime.

As mentioned in Section 1, an interesting example of a three-player quantum game was discussed by Vaidman [11, 12] who described the GHZ paradox [10] as a game among three players. Vaidman constructed a game, and tailored its winning conditions, such that the winning chances for a classical team of three players cannot exceed 75%. A team of quantum players, however, are able to win the game 100% if they share a GHZ state.

An analysis of Vaidman's game shows that it is won 100% by a team of quantum players having access to a probabilistic physical system for which the joint probabilities are non-factorizable in a way described by the winning conditions of the game. These conditions are constructed such that a set of non-factorizable joint probabilities generated by the GHZ state results in the team always winning the game.

Although Vaidman's game demonstrates how the GHZ state can be helpful in winning a game, by itself this quantum game does not present a quantization scheme for a general three-player noncooperative game. This was achieved by Benjamin and Hayden [18] who developed a multiplayer extension of Eisert et al.'s quantization scheme [17], originally proposed for two-player noncooperative games. Eisert et al.'s scheme is widely considered to have led to the birth of the area of quantum games.

However, Vaidman's game offers an interesting situation, which motivates one to ask what may happen to a generalized three-player noncooperative symmetric game, when the participating players share a probabilistic system for which joint probabilities are not factorizable. This is precisely the question that we aim to address in the present paper.

### 4.1 Three-player quantum games using EPR-Bohm setting

We consider an EPR-Bohm setting for three spatially-separated observers and use it to play a general three-player symmetric noncooperative game. This setting can be described as follows:

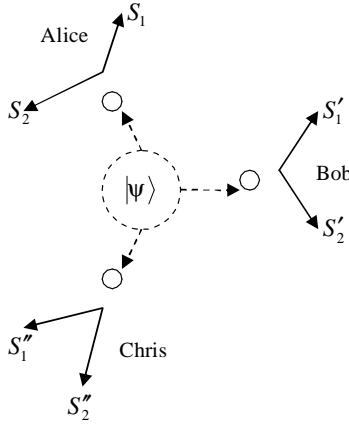


Figure 1: In the EPR-Bohm setting for playing a three-player quantum game, players Alice, Bob, and Chris each receive a particle, in a run, coming from a tripartite state. Each player has to decide one between the two available directions in the run and has to inform the referee of his/her choice. The referee makes a quantum measurement along the three chosen directions with Pauli spin operators. The players' payoff relations are made public at the start of the game and depend on the directions the players choose over a large number of runs (defining their strategies), the matrix of the game, and on the joint probabilities of the measurement outcomes that the referee obtains.

a) Three observers, henceforth called the players Alice, Bob, and Chris, are distantly located and are not able to communicate among themselves.

b) In a run, each player receives a particle and has to choose one out of the two directions and to inform the referee of his/her choice (henceforth called his/her pure strategy).

c) The referee, after receiving information on the three players' strategies in a run (which are three directions) rotates his Stern-Gerlach type detectors along these directions and makes a measurement using Pauli spin observables.

d) The two directions available to each player correspond to the two kinds of (non-commuting) measurements that can be performed by the referee in a run.

e) The outcome of a measurement, in a run, along any one of the three chosen directions is either  $+1$  or  $-1$ .

f) Over a large number of runs, a player can play a mixed strategy when s/he has a linear combination (with normalized and real coefficients) of choosing between the two available directions as the outcomes of quantum measurements for all runs are recorded by the referee.

It can be noticed that in this EPR-Bohm setting, instead of coins, we can let  $S_1$  and  $S_2$  to be Alice's two directions. Similarly, we can let  $S'_1$  and  $S'_2$  be Bob's two directions, and, similarly, we can let  $S''_1$  and  $S''_2$  be Chris' two directions. As the outcome of a quantum measurement along one of the three directions is  $+1$  or  $-1$  that we have associated to the Head or Tail state of a coin, respectively.

Recall that in our six-coin setup, in a run, each player has to choose one between two coins and to inform this choice to the referee. The referee, after knowing which three coins the players have chosen in that run, tosses them together and obtains a Head or Tail state for each.

Compare this to what happens in the EPR-Bohm setting in which, in a run, the outcome of a quantum measurement is  $+1$  or  $-1$ , along each one of three directions chosen by the players. Note that players' strategies in the EPR-Bohm setting are same as they are when they play classical strategies using six coins.

This definition of strategy serves three purposes: a) it allows, in a straight forward fashion, a quantum version of the three-player game from the three-player classical game that is played using six coins b) as the strategies remain identical in both the classical and the quantum games, it

seems that the Enk and Pike's argument [56] is avoided<sup>2</sup> c) it allows more direct exploitation of the peculiar non-factorizable quantum mechanical probabilities in terms of game-theoretic outcomes.

One may ask here about the quantum state(s) that are measured by the referee after the players inform him/her of their strategies. In fact, apart from the fact that the proposed setting uses three-partite quantum states, no further restrictions are placed telling what should be the input quantum states. That is, any pure or mixed three-partite states can be used to play the quantum game.

#### 4.1.1 Joint probabilities in EPR-Bohm setting

Using the above setting and noticing the notation for 64 six-coin joint probabilities in (26-33) convinces one to introduce the same notation for 64 joint probabilities corresponding to the EPR-Bohm setting for playing a three-player quantum game:

$$\begin{aligned}
p_1 &= \Pr(+1, +1, +1; S_1, S'_1, S''_1), & p_5 &= \Pr(-1, +1, +1; S_1, S'_1, S''_1), \\
p_2 &= \Pr(+1, -1, +1; S_1, S'_1, S''_1), & p_6 &= \Pr(-1, -1, +1; S_1, S'_1, S''_1), \\
p_3 &= \Pr(+1, +1, -1; S_1, S'_1, S''_1), & p_7 &= \Pr(-1, +1, -1; S_1, S'_1, S''_1), \\
p_4 &= \Pr(+1, -1, -1; S_1, S'_1, S''_1), & p_8 &= \Pr(-1, -1, -1; S_1, S'_1, S''_1);
\end{aligned} \tag{44}$$

$$\begin{aligned}
p_9 &= \Pr(+1, +1, +1; S_2, S'_1, S''_1), & p_{13} &= \Pr(-1, +1, +1; S_2, S'_1, S''_1), \\
p_{10} &= \Pr(+1, -1, +1; S_2, S'_1, S''_1), & p_{14} &= \Pr(-1, -1, +1; S_2, S'_1, S''_1), \\
p_{11} &= \Pr(+1, +1, -1; S_2, S'_1, S''_1), & p_{15} &= \Pr(-1, +1, -1; S_2, S'_1, S''_1), \\
p_{12} &= \Pr(+1, -1, -1; S_2, S'_1, S''_1), & p_{16} &= \Pr(-1, -1, -1; S_2, S'_1, S''_1);
\end{aligned} \tag{45}$$

$$\begin{aligned}
p_{17} &= \Pr(+1, +1, +1; S_1, S'_2, S''_1), & p_{21} &= \Pr(-1, +1, +1; S_1, S'_2, S''_1), \\
p_{18} &= \Pr(+1, -1, +1; S_1, S'_2, S''_1), & p_{22} &= \Pr(-1, -1, +1; S_1, S'_2, S''_1), \\
p_{19} &= \Pr(+1, +1, -1; S_1, S'_2, S''_1), & p_{23} &= \Pr(-1, +1, -1; S_1, S'_2, S''_1), \\
p_{20} &= \Pr(+1, -1, -1; S_1, S'_2, S''_1), & p_{24} &= \Pr(-1, -1, -1; S_1, S'_2, S''_1);
\end{aligned} \tag{46}$$

$$\begin{aligned}
p_{25} &= \Pr(+1, +1, +1; S_1, S'_1, S''_2), & p_{29} &= \Pr(-1, +1, +1; S_1, S'_1, S''_2), \\
p_{26} &= \Pr(+1, -1, +1; S_1, S'_1, S''_2), & p_{30} &= \Pr(-1, -1, +1; S_1, S'_1, S''_2), \\
p_{27} &= \Pr(+1, +1, -1; S_1, S'_1, S''_2), & p_{31} &= \Pr(-1, +1, -1; S_1, S'_1, S''_2), \\
p_{28} &= \Pr(+1, -1, -1; S_1, S'_1, S''_2), & p_{32} &= \Pr(-1, -1, -1; S_1, S'_1, S''_2);
\end{aligned} \tag{47}$$

$$\begin{aligned}
p_{33} &= \Pr(+1, +1, +1; S_1, S'_2, S''_2), & p_{37} &= \Pr(-1, +1, +1; S_1, S'_2, S''_2), \\
p_{34} &= \Pr(+1, -1, +1; S_1, S'_2, S''_2), & p_{38} &= \Pr(-1, -1, +1; S_1, S'_2, S''_2), \\
p_{35} &= \Pr(+1, +1, -1; S_1, S'_2, S''_2), & p_{39} &= \Pr(-1, +1, -1; S_1, S'_2, S''_2), \\
p_{36} &= \Pr(+1, -1, -1; S_1, S'_2, S''_2), & p_{40} &= \Pr(-1, -1, -1; S_1, S'_2, S''_2);
\end{aligned} \tag{48}$$

$$\begin{aligned}
p_{41} &= \Pr(+1, +1, +1; S_2, S'_1, S''_2), & p_{45} &= \Pr(-1, +1, +1; S_2, S'_1, S''_2), \\
p_{42} &= \Pr(+1, -1, +1; S_2, S'_1, S''_2), & p_{46} &= \Pr(-1, -1, +1; S_2, S'_1, S''_2), \\
p_{43} &= \Pr(+1, +1, -1; S_2, S'_1, S''_2), & p_{47} &= \Pr(-1, +1, -1; S_2, S'_1, S''_2), \\
p_{44} &= \Pr(+1, -1, -1; S_2, S'_1, S''_2), & p_{48} &= \Pr(-1, -1, -1; S_2, S'_1, S''_2);
\end{aligned} \tag{49}$$

$$\begin{aligned}
p_{49} &= \Pr(+1, +1, +1; S_2, S'_2, S''_1), & p_{53} &= \Pr(-1, +1, +1; S_2, S'_2, S''_1), \\
p_{50} &= \Pr(+1, -1, +1; S_2, S'_2, S''_1), & p_{54} &= \Pr(-1, -1, +1; S_2, S'_2, S''_1), \\
p_{51} &= \Pr(+1, +1, -1; S_2, S'_2, S''_1), & p_{55} &= \Pr(-1, +1, -1; S_2, S'_2, S''_1), \\
p_{52} &= \Pr(+1, -1, -1; S_2, S'_2, S''_1), & p_{56} &= \Pr(-1, -1, -1; S_2, S'_2, S''_1);
\end{aligned} \tag{50}$$

$$\begin{aligned}
p_{57} &= \Pr(+1, +1, +1; S_2, S'_2, S''_2), & p_{61} &= \Pr(-1, +1, +1; S_2, S'_2, S''_2), \\
p_{58} &= \Pr(+1, -1, +1; S_2, S'_2, S''_2), & p_{62} &= \Pr(-1, -1, +1; S_2, S'_2, S''_2), \\
p_{59} &= \Pr(+1, +1, -1; S_2, S'_2, S''_2), & p_{63} &= \Pr(-1, +1, -1; S_2, S'_2, S''_2), \\
p_{60} &= \Pr(+1, -1, -1; S_2, S'_2, S''_2), & p_{64} &= \Pr(-1, -1, -1; S_2, S'_2, S''_2).
\end{aligned} \tag{51}$$

Notice that for these coins the joint probabilities are factorizable and (44-51) reduce to (26-33).

<sup>2</sup>Essentially, this argument states that the process of allowing players extended sets of strategies, in a quantum game setup that is based on Eisert et al.'s scheme of playing a quantum game, can be equated to extending their sets of pure strategies in the classical game.

### 4.1.2 Constraints on joint probabilities

We note that quantum mechanics imposes constraints on the joint probabilities (44-51) that are usually known [51] as the normalization and the causal communication constraints. Using the definitions (44-51) the constraints imposed by normalization are:

$$\begin{aligned} \sum_{i=1}^8 p_i = 1, \quad \sum_{i=9}^{16} p_i = 1, \quad \sum_{i=17}^{24} p_i = 1, \quad \sum_{i=25}^{32} p_i = 1, \\ \sum_{i=33}^{40} p_i = 1, \quad \sum_{i=41}^{48} p_i = 1, \quad \sum_{i=49}^{56} p_i = 1, \quad \sum_{i=57}^{64} p_i = 1. \end{aligned} \quad (52)$$

The first equation in (52) with reference to the definitions (44) of the joint probabilities  $p_1, p_2, \dots, p_8$ , for example, describes the situation when Alice, Bob, and Chris play  $S_1, S'_1$ , and  $S''_1$ , respectively, over all runs. In this case the probabilities  $p_1, p_2, \dots, p_8$  correspond to the outcomes  $(+1, +1, +1), (+1, -1, +1), (+1, +1, -1), (+1, -1, -1), (-1, +1, +1), (-1, -1, +1), (-1, +1, -1)$ , and  $(-1, -1, -1)$ , respectively. As these are the only possible outcomes for the strategy triplet  $(S_1, S'_1, S''_1)$ , the corresponding probabilities must add up to one. The remaining seven equations in (52) can be interpreted similarly.

Notice that in six-coin setup the triplet  $(S_1, S'_1, S''_1)$  refers to the situation when Alice, Bob, and Chris go for the  $S_1$ -coin, the  $S'_1$ -coin, and the  $S''_1$ -coin, respectively, for all runs. The corresponding eight coin probabilities (26) are normalized and satisfy the first equation in (52). The same holds true for the remaining coin probabilities (27-33) as they satisfy the remaining equations in (52).

Along with the constraints that normalization imposes on joint probabilities, there are other constraints that are imposed by requirements of causal communication. Often these constraints are referred [51] to as the condition of ‘parameter independence’, ‘simple locality’, ‘signal locality’, or ‘physical locality’. Essentially, they say, for example, that the probability  $\Pr^A(+1; S_1)$  of Alice obtaining the outcome  $+1$  when she plays  $S_1$  is not changed by Bob’s choice for  $S'_1$  or  $S''_1$  and Chris’ choice for  $S'_1$  or  $S''_1$ . That is, the probability of a particular measurement outcome on one part of the system is independent of which kind of measurement is performed by the referee on the other part(s) [51]. Causal communication constraints make it impossible for the participating agents to acausally exchange the classical information.

The probability  $\Pr^A(+1; S_1)$  corresponds to when Alice chooses  $S_1$  and the referee obtains the outcome  $+1$  along  $S_1$ . Independence of Bob’s and Chris’ choices requires that,

$$\Pr^A(+1; S_1, S'_1, S''_1) = \Pr^A(+1; S_1, S'_1, S''_2) = \Pr^A(+1; S_1, S'_2, S''_1) = \Pr^A(+1; S_1, S'_2, S''_2), \quad (53)$$

which can be expanded using (44-51) as

$$\begin{aligned} \Pr(+1; S_1) &= \Pr(+1, +1, +1; S_1, S'_1, S''_1) + \Pr(+1, +1, -1; S_1, S'_1, S''_1) + \\ &\quad \Pr(+1, -1, +1; S_1, S'_1, S''_1) + \Pr(+1, -1, -1; S_1, S'_1, S''_1) \\ &= \Pr(+1, +1, +1; S_1, S'_1, S''_2) + \Pr(+1, +1, -1; S_1, S'_1, S''_2) + \\ &\quad \Pr(+1, -1, +1; S_1, S'_1, S''_2) + \Pr(+1, -1, -1; S_1, S'_1, S''_2) \\ &= \Pr(+1, +1, +1; S_1, S'_2, S''_1) + \Pr(+1, +1, -1; S_1, S'_2, S''_1) + \\ &\quad \Pr(+1, -1, +1; S_1, S'_2, S''_1) + \Pr(+1, -1, -1; S_1, S'_2, S''_1) \\ &= \Pr(+1, +1, +1; S_1, S'_2, S''_2) + \Pr(+1, +1, -1; S_1, S'_2, S''_2) + \\ &\quad \Pr(+1, -1, +1; S_1, S'_2, S''_2) + \Pr(+1, -1, -1; S_1, S'_2, S''_2). \end{aligned} \quad (54)$$

This reads as,

$$p_1 + p_2 + p_3 + p_4 = p_{25} + p_{26} + p_{27} + p_{28} = p_{17} + p_{18} + p_{19} + p_{20} = p_{33} + p_{34} + p_{35} + p_{36}. \quad (55)$$

Constraints similar to (54) can be written for the probabilities  $\Pr^A(-1; S_1), \Pr^A(+1; S_2), \Pr^A(-1; S_2)$  that correspond to Alice’s choices; for the probabilities  $\Pr^B(+1; S'_1), \Pr^B(-1; S'_1), \Pr^B(+1; S'_2), \Pr^B(-1; S'_2)$

that correspond to Bob's choices; and for the probabilities  $\overset{C}{\text{Pr}}(+1; S''_1)$ ,  $\overset{C}{\text{Pr}}(-1; S''_1)$ ,  $\overset{C}{\text{Pr}}(+1; S''_2)$ ,  $\overset{C}{\text{Pr}}(-1; S''_2)$  that correspond to Chris' choices.

Using the definitions (44-51) for Alice's choices these constraints read

$$\begin{aligned}
p_5 + p_6 + p_7 + p_8 &= p_{29} + p_{30} + p_{31} + p_{32} = p_{21} + p_{22} + p_{23} + p_{24} = p_{37} + p_{38} + p_{39} + p_{40}, \\
p_9 + p_{10} + p_{11} + p_{12} &= p_{41} + p_{42} + p_{43} + p_{44} = p_{49} + p_{50} + p_{51} + p_{52} = p_{57} + p_{58} + p_{59} + p_{60}, \\
p_{13} + p_{14} + p_{15} + p_{16} &= p_{45} + p_{46} + p_{47} + p_{48} = p_{53} + p_{54} + p_{55} + p_{56} = p_{61} + p_{62} + p_{63} + p_{64},
\end{aligned} \tag{56}$$

and for Bob's choices they read

$$\begin{aligned}
p_1 + p_3 + p_5 + p_7 &= p_{25} + p_{27} + p_{29} + p_{31} = p_9 + p_{11} + p_{13} + p_{15} = p_{41} + p_{43} + p_{45} + p_{47}, \\
p_2 + p_4 + p_6 + p_8 &= p_{26} + p_{28} + p_{30} + p_{32} = p_{10} + p_{12} + p_{14} + p_{16} = p_{42} + p_{44} + p_{46} + p_{48}, \\
p_{17} + p_{19} + p_{21} + p_{23} &= p_{33} + p_{35} + p_{37} + p_{39} = p_{49} + p_{51} + p_{53} + p_{55} = p_{57} + p_{59} + p_{61} + p_{63}, \\
p_{18} + p_{20} + p_{22} + p_{24} &= p_{34} + p_{36} + p_{38} + p_{40} = p_{50} + p_{52} + p_{54} + p_{56} = p_{58} + p_{60} + p_{62} + p_{64},
\end{aligned} \tag{57}$$

and for Chris' choices they read

$$\begin{aligned}
p_1 + p_2 + p_5 + p_6 &= p_{17} + p_{18} + p_{21} + p_{22} = p_9 + p_{10} + p_{13} + p_{14} = p_{49} + p_{50} + p_{53} + p_{54}, \\
p_3 + p_4 + p_7 + p_8 &= p_{19} + p_{20} + p_{23} + p_{24} = p_{11} + p_{12} + p_{15} + p_{16} = p_{51} + p_{52} + p_{55} + p_{56}, \\
p_{25} + p_{26} + p_{29} + p_{30} &= p_{33} + p_{34} + p_{37} + p_{38} = p_{41} + p_{42} + p_{45} + p_{46} = p_{57} + p_{58} + p_{61} + p_{62}, \\
p_{27} + p_{28} + p_{31} + p_{32} &= p_{35} + p_{36} + p_{39} + p_{40} = p_{43} + p_{44} + p_{47} + p_{48} = p_{59} + p_{60} + p_{63} + p_{64}.
\end{aligned} \tag{58}$$

Notice that the coin probabilities (26-33) satisfy the causal communication constraints (55-58) along with the constraints (52) imposed by normalization.

However, in an EPR-Bohm setting involving three observers, there may emerge a set of quantum mechanical joint probabilities (44-51) that cannot be expressed in the form of the joint probabilities (26-33), which correspond to six coins. In other words this means that for this set it is not possible to find six numbers  $r, s, r', s', r'', s'' \in [0, 1]$  such that the set (44-51) can be reproduced from the coin probabilities using (26-33).

## 4.2 Three-player quantum Prisoner's Dilemma

The set of joint probabilities in the six-coin setup is factorizable in terms of six probabilities  $r, s, r', s', r'', s''$  and the constraints (42) ensure that this set leads to the emergence of the strategy triplet  $(x^*, y^*, z^*) = (0, 0, 0)$  as a NE in three-player symmetric game of PD. Whereas a set of quantum mechanical joint probabilities that is obtained, for example, from three qubits in some quantum state, is not necessarily factorizable.

To ensure that the classical game and its classical solution remains embedded in the quantum game we require that the set of quantum mechanical joint probabilities (44-51) also satisfies the constraints (42). This is central to present argument as it guarantees that when the set (44-51) is factorizable the strategy triplet  $(D, D, D)$ , also represented as  $(x^*, y^*, z^*) = (0, 0, 0)$ , comes out to be the unique NE of the game. When the constraints (42) are fulfilled, the relations (52) describing normalization and the relations (56-58), which describe the causal communication constraint, are reduced to

$$\begin{aligned}
p_1 + p_2 + p_3 + p_4 + p_5 + p_6 + p_7 + p_8 &= 1, \\
p_{13} + p_{14} + p_{15} + p_{16} &= 1, \\
p_{18} + p_{20} + p_{22} + p_{24} &= 1, \\
p_{27} + p_{28} + p_{31} + p_{32} &= 1, \\
p_{36} + p_{40} &= 1, \quad p_{47} + p_{48} = 1, \\
p_{54} + p_{56} &= 1, \quad p_{64} = 1;
\end{aligned} \tag{59}$$

$$\begin{aligned}
p_1 + p_2 + p_3 + p_4 &= p_{27} + p_{28} = p_{18} + p_{20} = p_{36}, \\
p_5 + p_6 + p_7 + p_8 &= p_{31} + p_{32} = p_{22} + p_{24} = p_{40}, \\
p_{13} + p_{14} + p_{15} + p_{16} &= 1;
\end{aligned} \tag{60}$$

$$\begin{aligned}
p_1 + p_3 + p_5 + p_7 &= p_{27} + p_{31} = p_{13} + p_{15} = p_{47}, \\
p_2 + p_4 + p_6 + p_8 &= p_{28} + p_{32} = p_{14} + p_{16} = p_{48}, \\
p_{18} + p_{20} + p_{22} + p_{24} &= 1;
\end{aligned} \tag{61}$$

$$\begin{aligned}
p_1 + p_2 + p_5 + p_6 &= p_{18} + p_{22} = p_{13} + p_{14} = p_{54}, \\
p_3 + p_4 + p_7 + p_8 &= p_{20} + p_{24} = p_{15} + p_{16} = p_{56}, \\
p_{27} + p_{28} + p_{31} + p_{32} &= 1.
\end{aligned} \tag{62}$$

Requiring a set of 64 joint probabilities to satisfy the constraints (42) embeds the classical game within a quantum one as when the set becomes factorizable the strategy triplet  $(D, D, D) \equiv (0, 0, 0)$  emerges as a unique NE. Now, for PD the most interesting situation is when the strategy of Cooperation  $(C)$ , on the behalf of all the three players, comes out as a NE of the game—as this is also the Pareto-optimal [1] solution of the game. In our notation,  $(C, C, C)$  is given by  $(x^*, y^*, z^*) = (1, 1, 1)$ .

#### 4.2.1 Non-factorizable probabilities generating the pareto-optimal Nash equilibrium

We now explore whether the strategy profile  $(C, C, C)$  can be a NE for a set of non-factorizable, and thus quantum mechanical, joint probabilities. The explicit expressions, which state the conditions (23) for the strategy profile  $(C, C, C)$  to be a NE in PD, can be found as follows. Use the payoff relations (22) and (43) along with the constraints (59-62) to reduce the conditions (23) to

$$\begin{aligned}
\{p_5 + (\alpha/\beta)p_1 - p_{13}\} + (\theta/\beta)\{p_6 + p_7 - p_{14} - p_{15} + (\delta/\theta)(p_2 + p_3)\} + \\
(\omega/\beta)\{p_8 - p_{16} + (\epsilon/\omega)p_4\} \geq 0;
\end{aligned} \tag{63}$$

$$\begin{aligned}
\{p_2 + (\alpha/\beta)p_1 - p_{18}\} + (\theta/\beta)\{p_4 + p_6 - p_{20} - p_{22} + (\delta/\theta)(p_3 + p_5)\} + \\
(\omega/\beta)\{p_8 - p_{24} + (\epsilon/\omega)p_7\} \geq 0;
\end{aligned} \tag{64}$$

$$\begin{aligned}
\{p_3 + (\alpha/\beta)p_1 - p_{27}\} + (\theta/\beta)\{p_4 + p_7 - p_{28} - p_{31} + (\delta/\theta)(p_2 + p_5)\} + \\
(\omega/\beta)\{p_8 - p_{32} + (\epsilon/\omega)p_6\} \geq 0;
\end{aligned} \tag{65}$$

where the definition (10) of generalized three-player PD requires that each one of the five quantities  $\alpha/\beta, \theta/\beta, \delta/\theta, \omega/\beta, \epsilon/\omega$  are less than 1.

With this our question, whether the strategy profile  $(C, C, C)$  can exist as a NE in a symmetric three-player quantum game of PD, is now re-expressed in terms of finding five quantities  $\alpha/\beta, \theta/\beta, \delta/\theta, \omega/\beta, \epsilon/\omega$  (all less than 1) and a set of 64 (non-factorizable) joint probabilities for which the conditions (59, 60-62) as well as the constraints (42) hold. If the inequalities (63-65) hold for these five quantities and for the set of 64 (non-factorizable) joint probabilities, it will show that the strategy profile  $(C, C, C)$  becomes a NE in the quantum game.

It turns out that, in fact, it is not difficult to find such five quantities, all less than 1, and a set of 64 joint probabilities for which all of the above requirements hold. Consider the following example. Assign values  $\alpha/\beta = 9/10, \theta/\beta = 1/100, \delta/\theta = 1/5, \omega/\beta = 1/100,$  and  $\epsilon/\omega = 9/10$  for which the game becomes that of PD. Let the ten probabilities  $p_1, p_3, p_5, p_6, p_{13}, p_{15}, p_{18}, p_{20}, p_{22}, p_{27}$  be ‘independent’ and assign values to them as  $p_1 = 1/10, p_3 = 13/100, p_5 = 4/25, p_6 = 1/10, p_{13} = 7/50, p_{15} = 2/5, p_{18} = 13/100, p_{20} = 1/4, p_{22} = 37/100,$  and  $p_{27} = 1/5$ . Note that the constraints (42) assign zero value to thirty seven probabilities out of the remaining joint probabilities. Then the values assigned to the rest of (out of the total of 64) joint probabilities can be found from (59) and (60-62) as  $p_2 = 7/50, p_4 = 1/100, p_7 = 3/20, p_8 = 21/100, p_{14} = 9/25, p_{16} = 1/10, p_{24} = 1/4, p_{28} = 9/50, p_{31} = 17/50, p_{32} = 7/25, p_{36} = 19/50, p_{40} = 31/50, p_{47} = 27/50, p_{48} = 23/50, p_{54} = 1/2,$  and  $p_{56} = 1/2$ .

Now, for these values the left sides of the NE inequalities (63-65) are found as 106/1000, 96/1000, and 17/1000, respectively, showing that in this case the strategy profile  $(C, C, C)$ , consisting of the three players playing the strategy of Cooperation, is a NE. Note that this NE emerges because the resulting set of joint probabilities is non-factorizable and that the constraints (42) ensure that when this set becomes factorizable this NE disappears and  $(D, D, D)$  becomes the unique NE.

We notice that for a given set of numbers  $\alpha/\beta, \theta/\beta, \delta/\theta, \omega/\beta$  (each being less than 1) not every non-factorizable set of joint probabilities can result in the strategy triplet  $(C, C, C)$  being a NE. That is, in this framework non-factorizability is a necessary but not sufficient to make the strategy triplet  $(C, C, C)$  to be a NE. Also, in this paper we have not explored which other NE, apart from the strategy triplet  $(C, C, C)$ , may emerge for a given non-factorizable set of probabilities.

In the above approach considering three-player quantum game of PD we have assumed that for a given set<sup>3</sup> of joint probabilities, for which both the *normalization condition* and the *causal communication constraint* hold<sup>4</sup>, it is always possible in principle to find a three-party two-choice EPR-Bohm setting, with the relevant pure or mixed state(s) and appropriate directions of measurements for the three observers, which can always generate the given set of joint probabilities.

#### 4.2.2 Fate of the classical Nash equilibrium

For factorizable joint probabilities the strategy profile  $(D, D, D)$  is the unique NE, represented by the strategy triplet  $(x^*, y^*, z^*) = (0, 0, 0)$ . Here we show that this strategy profile remains a NE even when joint probabilities may become non-factorizable. Using the payoff relations (22) and (43) along with the constraints (59, 60-62), the NE conditions (23) for the strategy profile  $(D, D, D)$  for PD read

$$\begin{aligned}\Pi_A(0, 0, 0) - \Pi_A(x, 0, 0) &= -x(\epsilon p_{36} + \omega p_{40} - \omega), \\ \Pi_B(0, 0, 0) - \Pi_B(0, y, 0) &= -y(\epsilon p_{47} + \omega p_{48} - \omega), \\ \Pi_C(0, 0, 0) - \Pi_C(0, 0, z) &= -z(\epsilon p_{54} + \omega p_{56} - \omega).\end{aligned}\tag{66}$$

Now, from (59) we have  $p_{36} + p_{40} = 1$ ,  $p_{47} + p_{48} = 1$ , and  $p_{54} + p_{56} = 1$ . This makes the above three payoff differences to be  $x p_{36}(\omega - \epsilon)$ ,  $y p_{47}(\omega - \epsilon)$ , and  $z p_{54}(\omega - \epsilon)$  for Alice, Bob, and Chris, respectively. As for PD we have  $\omega > \epsilon$  which makes these quantities non-negative and the strategy profile  $(D, D, D)$  remains a NE even when the joint probabilities become non-factorizable. That is, a set of non-factorizable joint probabilities can only add to the classical NE of  $(D, D, D)$ .

## 5 Discussion

This paper extends our probabilistic framework for playing two-player quantum games to the multiplayer case. This framework presents an argument for the construction of quantum games without referring to the tools of quantum mechanics, thus making this area more accessible to workers outside the quantum physics discipline.

Where does this framework stand in front of Enk and Pike's criticism [56], which refers to the standard setting for playing a quantum game using Eisert et al.'s formalism? Their criticism considers a quantum game equivalent to playing another classical game in which players have access to extended set of classical strategies. This paper uses EPR-Bohm setting for playing quantum games in which each player has two directions, along which a measurement can be made by the referee, and his/her pure strategy, in a run, consists of choosing one direction. A mixed strategy is then defined to be the probability of playing one of his/her pure strategies. As the sets of strategies remain exactly identical in both the classical and the quantum forms of the game, it difficult to construct an Enk and Pike type argument [56] for a quantum game played in the EPR-Bohm setting.

<sup>3</sup>It may, however, not be known whether such a set has been obtained classically or quantum mechanically.

<sup>4</sup>as described by Cereceda [51].

In the literature the wording ‘quantum games’ has been used in many different contexts, which involve games and quantum settings in one way or the other. In view of this the present paper comes in line with the approach towards quantum games that originated with Eisert et al.’s quantization of Prisoner’s Dilemma [17], where they proposed a general procedure to quantize a given noncooperative two-player game. This approach is distinct, and is to be contrasted, from other approaches that also use the wording ‘quantum game’, in that it is placed in the context of classical game theory with the use of the Nash equilibrium concept. For example, in some approaches both ‘the game’ and its ‘winning condition(s)’ are arbitrarily defined, tailored or constructed, in order to show that only using a quantum-mechanical implementation the winning condition(s) can be fulfilled. In contrast, this paper begins by defining payoff relations, instead of the ‘winning condition(s)’, and then finds how game-theoretic outcome(s) of a noncooperative game may change in relation to the quantum mechanical aspects of a probabilistic physical system, which the players share in order to play the game.

The approach followed in this paper establishes a relationship between the ‘classicality’ of the physical system (expressed by the joint probabilities being factorizable) and a ‘classical game’, in the sense that using a classical system to play a game results in the classical game. Establishing this relationship allows us in the following step to find how non-classical (thus quantum) behavior of the physical system (expressed by the joint probabilities being non-factorizable) may change the outcome(s) of the game.

This paper considers very unusual non-factorizable joint probabilities, which may emerge from an entangled quantum state, by putting forward three- and six-coin setups in order to play a three-player symmetric non-cooperative game. In the six-coin setup, players choose coins in each run, not necessarily unbiased, which are subsequently tossed and the outcome of each toss is observed. This setup translates playing of the game in terms of 64 joint probabilities. In the following step, this translation of the game allows us to consider the corresponding quantum game by bringing in the same number of joint probabilities, which now may not be factorizable. We then consider how these quantum mechanical probabilities may change the Nash equilibria of the game under the constraint that factorizable joint probabilities must lead to the classical solution of the game.

We achieve this by re-expressing players’ payoffs in terms of 64 joint probabilities  $p_i$ , the players’ strategies  $x, y, z$  and the coefficients defining the game. We then use Nash inequalities to find the equilibria. We find constraints on  $p_i$  which ensure that for factorizable  $p_i$  the game gives the classical outcome and thus it becomes interpretable as a classical mixed-strategy game. This is carried out by playing the game in the six-coin setup and using Nash inequalities to obtain constraints on the coin probabilities  $r, s, r', s', r'', s''$ , which reproduce the outcome of the classical mixed-strategy game. We use the relations (26-33), resulting from  $p_i$  being factorizable, to translate the constraints on  $r, s, r', s', r'', s''$  in terms of constraints on  $p_i$ . We refer to the standard three-party EPR-Bohm setup and allow  $p_i$  to be non-factorizable, while retaining the constraints on  $p_i$ . We then ask if non-factorizability may lead to the emergence of new solution(s) of the game. In case non-factorizability leads to new solution(s), given as a triplet(s)  $(x^*, y^*, z^*)$ , there is always a set of 64 non-factorizable joint probabilities that are associated with it.

Our results show that non-factorizability leads to a new NE for three-player PD that is pareto-optimal. As we recall from the Ref. [43] that this does not turn out to be the case for the two-player PD and the classical NE remains intact even when the players are given access to non-factorizable probabilities. There exist, however, examples of two-player games for which non-factorizability indeed leads to new NE. The game of chicken [1] is one such example. It seems that from a game-theoretic perspective the tri-partite quantum correlations are different and stronger relative to the bi-partite quantum correlations. Recall that non-factorizability, as it is defined above, is only a necessary, but not sufficient, condition for the violation of Bell’s inequality. That is, a non-factorizable set of joint probabilities may not violate Bell’s inequality but a set of joint probabilities that violates Bell’s inequality must be non-factorizable. This might motivate one to ask about the exact connection between non-factorizability, entanglement, and the violation of Bell’s inequality. However, non-factorizability being only a necessary condition for the violation of Bell’s inequality seems to go in line with the known result that a separable mixed state can

violate Bell's inequality.

The probabilistic framework towards quantum games, developed in Ref. [43] and extended to multiplayer games in this paper, uses the classical concept of probability, which is well known to be more restrictive than the quantum notion [57]. Essentially, the classical concept describes 'probability' as being a number between zero and one and that for a joint probability of two events this number is less or equal to the numbers corresponding to each of the events. Though being more restrictive, as Pitowski [57] describes it, this concept is "nevertheless rooted in some very basic intuition." If quantum games can be expressed in terms of this basic concept this can only be helpful to introduce this area to researchers outside the quantum domain.

It is relevant here to point out that the probabilistic framework for quantum games, proposed originally for two-player games in Ref. [43] and extended to three-player games in the present paper, appears close to Einstein's statistical interpretation of quantum mechanics [58, 59]. The key assertion of this interpretation describes a quantum state (pure or otherwise), representing an ensemble of similarly prepared systems, and need not provide a complete description of an individual system. By using coin tosses in order to translate playing of a classical game in terms of joint probabilities and subsequently introducing unusual quantum mechanical non-factorizable joint probabilities, the suggested framework uses the concept of an ensemble of similarly prepared systems. Multiple coin tosses, which are central to the present framework, are found helpful in understanding how quantum mechanical predictions in quantum game theory do not pertain to a single measurement, but relate to an ensemble of similar measurements.

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