

Altruistic Duality in Evolutionary Game Theory

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A game-theoretic dynamical model of social preference and enlightened self-interest is formulated. Existence of symmetry and duality in the game matrices with altruistic social preference is revealed. The model quantitatively describes the dynamical evolution of altruism in prisoner's dilemma and the regime change in prey-predator dynamics.

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Through the modeling of ecosystems, evolutionary game theory brings such diverse fields as biology, ecology, economics and sociology under the umbrella of mathematical sciences [1–3]. One central objective of evolutionary game theory is to understand the workings of cooperative behavior among the individuals in an ecosystem. Since the publication of the work by Axelrot on the prisoner's dilemma [4], it is generally understood that the concept of altruism holds a key to the emergence of cooperative behavior. While egoism, defined as the drive toward maximization of individual payoffs, is a cornerstone of game theory, casual observation reveals that altruism is just as universal a feature as egoism in systems consisting of like individuals. Altruism has obtained mathematical expression in the work of Bester and Güth [5], in which altruistic behavior is shown to become evolutionarily stable in certain situations through the enhancement of fitness of a majority of individuals in a system. While their mathematical treatment is general and elegant, it is formulated in static and descriptive language. Bringing in *dynamics* to the model would give it more predictive power.

In this article, we do not try to explain the emergence of altruistic cooperation. Rather, we intend to develop a game-theoretic model of ecosystem whose evolution is driven by the development of an optimal degree of altruism. Toward this goal, the separation of two time scales, one for the fast variation of dynamical variables, and the other for the secular variation of “environmental” coefficients, proves to be crucial [7]. The formulation of our model in terms of parametric game matrices reveals symmetry properties of altruistic game theory. We demonstrate the usefulness of our approach through numerical analyses of a game of prisoner's dilemma and a prey-predator system. In the latter example, we point out the existence of a regime change phenomenon caused by dynamical symmetry breaking.

We start by considering a system of N identical individuals randomly paired to repeatedly play a two-player game with $M+1$ pure strategies. We introduce the payoff

matrix A and the system average strategy vector \mathbf{x} ,

$$\begin{aligned} A &= \{A_{ij}\} \quad (i, j = 0, \dots, M), \\ \mathbf{x} &= \{x_i\} \quad (i = 0, \dots, M). \end{aligned} \quad (1)$$

All entries in A and \mathbf{x} are real numbers, and the relation $x_0 = 1 - x_1 - \dots - x_M$ is imposed. It is convenient to consider \mathbf{x} as column vector so that the matrix product $A\mathbf{x}$ is again a column vector. We index the elements of A such that a player with i -th strategy playing against another player with j -th strategy will obtain the payoff $A_{i,j}$. We interpret \mathbf{x} either as the system being made up of $x_i N$ players playing i -th strategy, or alternatively as N individuals adopting identical mixed strategies in which the probability of playing i -th strategy is given by x_i . A player with a mixed strategy specified by a vector \mathbf{s} in the system obtains the payoff

$$\langle \mathbf{s} | A | \mathbf{x} \rangle = \mathbf{s}^\dagger A \mathbf{x} = \mathbf{s}^\dagger \mathbf{p}(A, \mathbf{x}). \quad (2)$$

In the second equality, the payoff vector is defined as $\mathbf{p}(A, \mathbf{x}) \equiv A\mathbf{x}$, whose i -th element represents the payoff of a player with i -th pure strategy. We can average $\langle \mathbf{s} | A | \mathbf{x} \rangle$ over the entire system by the identification $\mathbf{s} = \mathbf{x}$, and obtain the average per capita payoff of the system $\Pi(\mathbf{x}) \equiv \langle \mathbf{x} | A | \mathbf{x} \rangle$. In spite of the use of bra-ket notations $\langle \mathbf{s} |$ and $| \mathbf{x} \rangle$, an obvious adoption from the quantum mechanics, all entries to the vectors are real numbers representing probabilities, and there should be no confusion on the fact that we are dealing with *classical* game theory.

As is well known, the best strategy for a game among players seeking immediate individual payoff maximization is given by the mixed Nash equilibrium [6] of the matrix A . This equilibrium, however, does not always give maximization of the system average payoff. A system consisting of players with longer view on their payoff often has a higher average payoff than a Hobbsian system consisting of narrowly egoistic players. To describe such “enlightened self-interest”, we follow Bester and Güth [5], and separate the process of reproduction from that of selection: The players switch strategies in pursuit of the maximization of a range of *perceived* payoffs which are related, but not necessarily identical to the real payoff. The deviation of perceived from real payoffs could represent imperfect information, socially imposed norm, or just error and caprice. The system is then assumed to

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be under slow selection process during which the players with inferior (real) payoff are pruned off. To formulate such two-stage evolution in a simple manner, we define a one-parameter family of matrix A^κ , which we call the *game matrix* with the *preference parameter* κ . We assume that A^κ reduces to the original payoff matrix A with $\kappa = 0$. The specific example we study is the game matrix

$$A^\kappa \equiv (1 - \kappa)A + \kappa A^\dagger, \quad \kappa \in [0, 1], \quad (3)$$

where A^\dagger is the transposed matrix of A defined by

$$A^\dagger = \{A_{ij}^\dagger\} \quad (i, j = 0, \dots, M); \quad A_{ij}^\dagger \equiv A_{ji}. \quad (4)$$

Notice that, in all instances in this article, superscript κ on game matrix A and strategy vector \mathbf{x} signify the preference parameter, *not* the exponentiation. Be prepared to see the notations $A^0 = A$, $A^1 = A^\dagger$ etc.. The meaning of A^\dagger becomes evident by considering its payoff vector $\mathbf{p}(A^\dagger, \mathbf{x}) = A^\dagger \mathbf{x}$, whose i -th component is the average payoff yielded to the opponent by a player with the pure strategy i . Therefore, we call A^\dagger the *altruistic dual matrix* of the original payoff matrix A . Obviously we have a relation $\langle \mathbf{x} | A | \mathbf{x} \rangle = \langle \mathbf{x} | A^\dagger | \mathbf{x} \rangle$, which simply means that per capita system payoff can be calculated from the payoff obtained by players, or from the payoff contributed by him to the rest of the system. We can generalize this result for A^κ as $A^{\kappa\dagger} = A^{1-\kappa}$, namely $A^{1-\kappa}$ is the altruistic dual matrix of A^κ . Per capita system payoff $\Pi(\mathbf{x})$ can be calculated from A^κ with any allowed value of κ ;

$$\Pi(\mathbf{x}) = \langle \mathbf{x} | A^\kappa | \mathbf{x} \rangle, \quad \kappa \in [0, 1]. \quad (5)$$

Consider a system with a given preference parameter κ which evolves with *replicator dynamics* [8], in which a player with strategy \mathbf{s} tinkers with the changes to the strategy by $\delta \mathbf{s}$, and accepts the changes in a probability proportional to the resulting gain in the payoff $\langle \mathbf{s} + \delta \mathbf{s} | A^\kappa | \mathbf{x} \rangle - \langle \mathbf{s} | A^\kappa | \mathbf{x} \rangle$. The time development of strategy \mathbf{x}^κ is described by the M -dimensional Lotka-Volterra equation

$$\begin{aligned} \frac{\dot{x}_i^\kappa}{x_i^\kappa} &= \partial_{s_i} \langle \mathbf{s} | A^\kappa | \mathbf{x}^\kappa \rangle \Big|_{\mathbf{s}=\mathbf{x}^\kappa} \\ &= p_i(A^\kappa, \mathbf{x}^\kappa) - p_0(A^\kappa, \mathbf{x}^\kappa), \end{aligned} \quad (6)$$

with $i = 1, \dots, M$. We refer to this dynamics as an A^κ -constrained game, or simply an A^κ -game. Typically, after some time period, \mathbf{x}^κ approaches a stable fixed point \mathbf{X}^κ , which we call A^κ -Nash equilibrium, that is obtained by the linear equation

$$p_i(A^\kappa, \mathbf{X}^\kappa) = p_0(A^\kappa, \mathbf{X}^\kappa), \quad (i = 1, \dots, M). \quad (7)$$

Suppose we have an *ensemble* of systems with various values of preference parameter κ . If there is a selection process based on the payoff $\Pi(\mathbf{X}^\kappa)$ at work, the average preference parameter κ shall evolve toward $\kappa = \kappa_{max}$ that

gives the maximum per capita system payoff $\Pi(X^{\kappa_{max}})$. For example, we can postulate

$$m \frac{\dot{\kappa}}{\kappa} = \partial_\kappa \langle \mathbf{X}^\kappa | A | \mathbf{X}^\kappa \rangle, \quad (8)$$

where m is a large number $m \gg 1$ that ensures slow secular variation of κ in comparison to the variation of dynamical variable \mathbf{x}^κ . We might alternatively consider the development of κ by Newtonian dynamics, in which case $-\Pi(\mathbf{X}^\kappa)$ should be identified as the potential.

Our task is reduced to evaluating the functional profile of $\Pi(\mathbf{X}^\kappa)$. Let us note the relation $\Pi(\mathbf{X}^\kappa) = p_i(A^\kappa, \mathbf{X}^\kappa)$ for arbitrary i , which is obtained from (5) and (7). Combining this with another equality

$$\begin{aligned} p_k(A^\kappa, \mathbf{X}^\kappa) &= \sum_i X_i^{1-\kappa} \sum_j A_{ij}^\kappa X_j^\kappa \\ &= \sum_i X_i^\kappa \sum_j A_{ij}^{1-\kappa} X_j^{1-\kappa} \\ &= p_l(A^{1-\kappa}, \mathbf{X}^{1-\kappa}), \end{aligned} \quad (9)$$

which is valid for arbitrary k and l , we obtain the *altruistic duality*

$$\Pi(\mathbf{X}^\kappa) = \Pi(\mathbf{X}^{1-\kappa}). \quad (10)$$

Namely, the per capita system payoff for an A^κ -game is exactly equal to that of its dual game $A^{1-\kappa}$. Specifically, we have $\Pi(\mathbf{X}^0) = \Pi(\mathbf{X}^1)$, an equivalence of payoff between a completely egoistic game A and a completely altruistic game A^\dagger . We stress that this duality is non-trivial, unlike, for example, the mere matrix symmetry (5). One immediate result of (10) is that $\kappa = 1/2$ has to be an extremum of the payoff $\Pi(\mathbf{X}^\kappa)$. If this is the sole maximum, we have the inequality

$$\Pi(\mathbf{X}^{1/2}) \geq \Pi(\mathbf{X}^\kappa) \geq \Pi(\mathbf{X}^0). \quad (11)$$

In general, there could be other extrema and also $\kappa = 1/2$ could be a minimum. But we shall show in the following examples, that there are indeed interesting cases in which (11) holds, and that examples include the classic prisoner's dilemma. The *maximum happiness of the maximum majority* is achieved when every individual in the system is constrained to pursue an equal mixing of egoistic and altruistic payoff. In hindsight, this is to be expected, since direct maximization of the per capita system payoff results in symmetrization of the game matrix A ;

$$\partial_{x_i} \langle \mathbf{x} | A | \mathbf{x} \rangle = \partial_{s_i} \langle \mathbf{s} | (A + A^\dagger) | \mathbf{x} \rangle \Big|_{\mathbf{s}=\mathbf{x}} = 0. \quad (12)$$

We illustrate our results with two examples. First, consider a two-strategy ($i = 0, 1$) game whose payoffs are specified by the matrix

$$A = \begin{pmatrix} 0 & \beta + \alpha \\ -\gamma & \beta \end{pmatrix}, \quad (13)$$

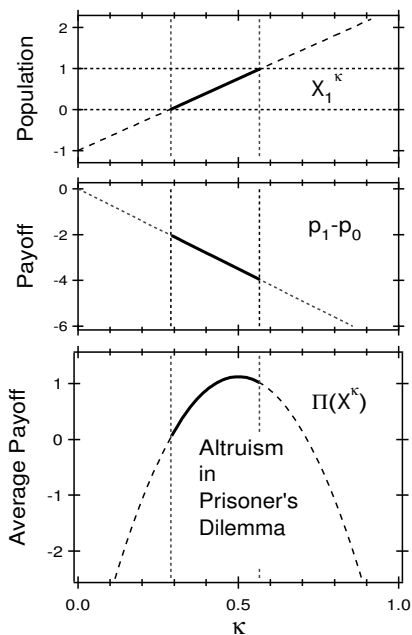


FIG. 1: The A^κ -constrained prisoner's dilemma specified by the payoff table (13). The parameters are chosen to be $\alpha = 4$, $\beta = 1$ and $\gamma = 2$. In the middle graph, p_i stands for $p_i(A, \mathbf{X}^\kappa)$.

where α , β and γ are positive real numbers that satisfy the condition $\alpha > \gamma$. This is the famous example of prisoner's dilemma: When two players show a "good hand", $i = 1$, both obtain the payoff of β , but when one player betrays the other by playing a bad hand, $i = 0$, he gets the Devil's reward of $\alpha + \beta$ while imposing the damage $-\gamma$ on the opponent. When both players show a "bad hand" there is no payoff. Temporal evolution of A^κ -constrained game is described by the logistic equation

$$\dot{x}_1^\kappa = [\kappa(\alpha + \beta + \gamma) - \gamma]x_1^\kappa - (\alpha - \gamma)(x_1^\kappa)^2. \quad (14)$$

The evolutionary Nash equilibrium is fixed point X_1^κ with an average payoff $\Pi(\mathbf{X}^\kappa)$ given respectively by

$$X_1^\kappa = -\frac{\gamma}{\alpha - \gamma} + \frac{\alpha + \beta + \gamma}{\alpha - \gamma}\kappa, \quad (15)$$

$$\Pi(\mathbf{X}^\kappa) = -\frac{(\alpha + \beta)\gamma}{\alpha - \gamma} + \frac{(\alpha + \beta + \gamma)^2}{\alpha - \gamma}\kappa(1 - \kappa).$$

The per capita system payoff $\Pi(\mathbf{X}^\kappa)$ is indeed reflection-symmetric with respect to the line $\kappa = 1/2$. An interesting quantity to look at is the difference $p_1(A, \mathbf{X}^\kappa) - p_0(A, \mathbf{X}^\kappa) = -(\alpha + \beta + \gamma)\kappa$. This is the payoff disparity between good and bad hands, which has to be tolerated by good-hand players to achieve an A^κ -game with a non-zero value of κ . The peculiarity of this game is that both for a purely egoistic game A^0 and for a purely altruistic game A^1 , the mixed strategy Nash equilibrium (15) is located outside of a realizable domain $0 \leq X_1 \leq 1$. Easy calculation shows that only $\kappa_0 \leq \kappa \leq \kappa_1$ with $\kappa_0 = \gamma/(\alpha + \beta + \gamma)$ and $\kappa_1 = \alpha/(\alpha + \beta + \gamma)$ is allowed.

These are the values that give $X_1^{\kappa_0} = 0$ and $X_1^{\kappa_1} = 1$. If $\kappa = 1/2$ falls between them, the system eventually reaches this optimum state having

$$X_1^{1/2} = \frac{\alpha + \beta - \gamma}{2(\alpha - \gamma)}, \quad \Pi(\mathbf{X}^{1/2}) = \frac{(\alpha + \beta - \gamma)^2}{4(\alpha - \gamma)}. \quad (16)$$

Otherwise, the system settles for $\Pi(\mathbf{X}^{\kappa_1}) = \beta$. Figure 1 depicts an example of the former case. We expect experimental studies to be performed to check these predictions. Also, comparison with numerical simulations with real strategies with memory (such as Tit-for-Tat or Pavlov) [4, 9–11] would be beneficial.

As our second example, we consider the following three-strategy ($i = 0, 1, 2$) game;

$$A = \begin{pmatrix} 0 & 0 & 0 \\ b & b - a & b - \rho \\ -d & f\rho - d & -d \end{pmatrix}, \quad (17)$$

where a , b , d , f and ρ are positive real numbers. With strategy 0, a player abstains. With strategy 1, he/she produces worth valued at b . Worth is reduced by a when the opponent also produces worth because of overcrowding. With strategy 2, the player wastes his/her resources valued at d . But he/she can derive worth valued at $f\rho$ from the worth-producing opponent by way of raiding and diminishing the opponent's worth by ρ . If we describe the system by three-strategy vector \mathbf{x} , a natural interpretation is that x_1 represents the portion of total population which subsists on environmental riches ("commoners"), and x_2 , the portion which tries to dominate the opponents ("knights"). Under the replicator dynamics with the A^κ -constrained game, the evolution of the system is governed by

$$\dot{x}_1^\kappa = (1 - \kappa)bx_1^\kappa - a(x_1^\kappa)^2 - (1 - \kappa - \kappa f)\rho x_1^\kappa x_2^\kappa \quad (18)$$

$$\dot{x}_2^\kappa = -(1 - \kappa)dx_2^\kappa + (f - \kappa - \kappa f)\rho x_1^\kappa x_2^\kappa.$$

This is the classical Lotka-Volterra prey-predator system, of which the Nash equilibrium is obtained as

$$X_1^\kappa = \frac{(1 - \kappa)d}{(f - \kappa - \kappa f)\rho}, \quad (19)$$

$$X_2^\kappa = \frac{(1 - \kappa)}{(1 - \kappa - \kappa f)\rho} \left[b - \frac{ad}{(f - \kappa - \kappa f)\rho} \right].$$

One complication is that, when κ becomes larger than $\kappa^* \equiv (f - ad/b\rho)/(1 + f)$, the fixed point X_2^κ falls below zero and becomes unstable. Concurrently, however, there appears a new trivial Nash equilibrium, which is given by

$$X_1^\kappa = \frac{(1 - \kappa)b}{a}, \quad X_2^\kappa = 0. \quad (20)$$

This case corresponds to a single-species logistic evolution, for which the game matrix is effectively reduced to

$$A^\kappa = \begin{pmatrix} 0 & \kappa b \\ (1 - \kappa)b & b - a \end{pmatrix}. \quad (21)$$

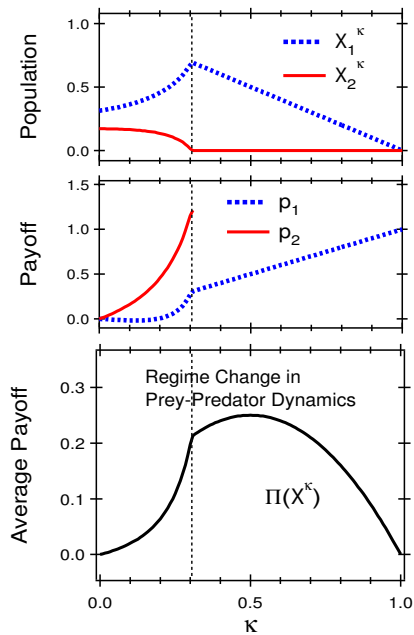


FIG. 2: The three-strategy game specified by the payoff matrix (17) with A^κ -constrained game dynamics. The parameters are chosen to be $a = 1$, $b = 1$, $d = 1$, $f = 0.8$ and $\rho = 4$. In the middle graph, p_i stands for $p_i(A, \mathbf{X}^\kappa)$, and p_0 is identically zero. At $\kappa = \kappa^*$, the system displays the transition from prey-predator dynamics to single species logistic dynamics.

In either case, the A^κ -Nash equilibrium yields the per capita system payoff

$$\Pi(\mathbf{X}^\kappa) = \frac{D}{T}; \quad (22)$$

$$D \equiv \det(A_{ij}^\kappa), \quad T \equiv \sum_{i,j} (-1)^{i+j} \Delta_{ij}(A^\kappa),$$

where $\Delta_{ij}(A^\kappa)$ is the minors of A^κ .

An example of this model with specific parameters is depicted in Figure 2. The $\Pi(\mathbf{X}^\kappa)$ has a single peak at $\kappa = 1/2$ as before, but because of the change in the nature of dynamics at κ^* , it is no longer symmetric. Suppose we start from a knight-commoner dynamics of $\kappa = 0$ game. The system is Pareto optimal in that payoffs for both commoners and knights are identical. Turning on non-zero κ amounts to introducing “altruistic culture”. Curiously, this reduces the population capacity of knights and, at the same time, increase its payoff. For commoners, it results in larger population and lower relative payoff. Although the overall per capita payoff increases quickly, the class disparity also increases with knights commanding ever higher relative payoff as the system becomes more altruistic. At $\kappa = \kappa^*$ however, the population capacity for knights become zero and the “aristocratic” regime collapses. Above κ^* , we have a “democratic” regime consisting of single self-sustaining population of commoners, ever prospering with increasing altruism until the system hits the ceiling level at $\kappa = 1/2$, which is the global stability point.

In both of the above two examples, $\kappa = 1/2$ turns out to be the only maximum in the allowed region. But in general, higher polynomials in (22) for larger M could result in more structures in the $\Pi(\mathbf{X}^\kappa)$ profile for higher numbers of strategies. We should thus expect to find more complex dynamics.

The indirect payoff maximization through the “communal” arrangement of social goods is widespread among ecosystems in which components have intellectual capacity. The examples we have studied in this article are simple toy models which do not necessarily have specific real-world counterparts. The fact that they show features reminiscent to concepts devised by the socio-economic philosophers in past centuries is rather intriguing.

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