

Equivalence of Local and Separable Realizations of the Discontinuity-Inducing Contact Interaction and Its Perturbative Renormalizability

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We prove that the separable and local approximations of the discontinuity-inducing zero-range interaction in one-dimensional quantum mechanics are equivalent. We further show that the interaction allows the perturbative treatment through the coupling renormalization.

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I. INTRODUCTION

The contact force different from the δ -function interaction is an object that appears to have defied the intuition since the recognition of its existence more than a decade ago [1–4]. This second kind of contact interaction is an object found only in the quantum mechanics of one-dimensional particles. It is characterized by the discontinuity of *wave functions themselves* at the location of the interaction which is contrasted to the discontinuity of the derivative of wave functions present in the “usual” δ function potentials. From its start, this second kind of contact interaction has been conceived in abstract mathematical settings which are rather detached from any particular physical model. Because of the barrier of arcane mathematical language, it has been mostly hidden from the view of the physics community at large.

There are some welcome signs which indicate a change in this situation. One is the recent discoveries of several non-trivial properties of the quantum systems with discontinuity-inducing interactions [5–8]. Also, there have been several attempts to represent the interaction in terms of physically realizable potential models [9,10]. Here, we single out one example in which the discontinuity-inducing contact interaction is constructed in terms of elementary local self-adjoint operator (“epsilon” potential) [11]. Its potential significance is in the fact that it can lead to the experimental manufacturing of wave function discontinuity and Neumann boundary. It appears that we are now ready for the application of the second kind of contact interaction to quantum system with practical relevance, in particular to the many-body systems [8]. One stumbling block to this end is the apparent non-perturbative nature of the interaction which can be recognized immediately by observing the divergence of the matrix element of the epsilon potential at the zero-size limit. None of the advanced field theoretical technique would be available without the representation of the interaction in the second-quantized form.

Interestingly, we realize that the earliest physical realization of the discontinuity-inducing interaction by Šeba as a separable potential with momentum dependent form factor (“prime-delta-prime” function) [3] is of a great relevance, since it rests on the concept of renormalization with the use of a coupling constant which disappears at the zero-range limit. It is evident at this point, that the clarification of the relation between “epsilon” and “prime-delta-prime” representations is called for. Also, it is helpful to have a closer look at the workings of the discontinuity-inducing interaction in the context of the perturbative treatment.

These two matters are exactly what we address in this paper. We first prove that the local and separable approximations of the discontinuity-inducing zero-range force in one-dimensional quantum mechanics are equivalent. We then look at the example of the spectra of a particle on a line with a contact force. One observes that the use of renormalized coupling results in the order-by-order cancellation of divergent term, and allows the perturbative treatment of the problem.

It is worthwhile to recall the zero-range limit of small obstacle quantum mechanics in other dimensions than one. As is well known, something resembling to δ -function can be defined only in weaker sense with renormalized couplings [4,12]. In hindsight, the fact that the simple δ -function limit exists for small size obstacle in one-dimensional quantum mechanics is an accident, which could be either seen as lucky or unlucky. The former view needs no explanation. The latter view seems equally plausible since this has in effect delayed the wide recognition of the discontinuity-inducing interaction as an indispensable element of the one-dimensional quantum mechanics. The situation needs to be rectified. We would feel our purpose fulfilled if the current work serves as a step stone to that road.

This paper is organized as follows. In the next Section, we present an elementary rederivation of the result of Šeba that the discontinuity-inducing interaction

can be expressed as a momentum dependent separable form. In Section III, the equivalence of Šeba's expression and the local expression is shown. Two coupling constants, bare and renormalized, are introduced. In Section IV, it is shown that the perturbative treatment of the discontinuity-inducing interaction brings about the divergence at the level of second order and beyond. This divergence, however, is shown to be made manageable with the coupling constant renormalization.

II. REDERIVATION OF SEBA'S PRIME-DELTA-PRIME INTERACTION

The discussion on discontinuity-inducing interaction traditionally started with the mathematical theory of self-adjoint extension. Here, we take another route which requires nothing fancier than the concept of δ -function as the zero-size limit of regular potential with constant volume integral.

We start by defining a function

$$\Delta_a(x) = \begin{cases} \frac{1}{2a} & |x| \leq a \\ 0 & |x| > a, \end{cases} \quad (1)$$

whose values at $x = \pm a$ are defined as the limiting values from $|x| < 0$ region. It has the property

$$\int_{-\infty}^{\infty} dx \Delta_a(x) = 1. \quad (2)$$

Obviously, one has the Dirac's δ -function as the zero-range limit;

$$\Delta_a(x) \rightarrow \delta(x) \quad (a \rightarrow 0). \quad (3)$$

From the relation

$$(\Delta_a(x))^n = \left(\frac{1}{2a}\right)^{n-1} \Delta_a(x), \quad (4)$$

one has

$$\Delta_a(x) f(\Delta_a(x)) = \Delta_a(x) f\left(\frac{1}{2a}\right). \quad (5)$$

We consider a wave function $\phi(x)$ that satisfies

$$-\frac{d^2}{dx^2}\phi(x) + v\Delta_a(x)\phi(x) = k^2\phi(x). \quad (6)$$

At $a \rightarrow 0$, this wave function obviously satisfies the connection condition

$$\phi'(0_+) - \phi'(0_-) = v\phi(0_+) = v\phi(0_-). \quad (7)$$

We now define another wave function $\psi(x)$ by

$$\psi(x) \equiv \frac{d}{dx}\phi(x), \quad (8)$$

and look for the wave equation for $\psi(x)$, and study its properties. From Eqs. (6) and (8), we have

$$\phi(x) = -\left(\frac{1}{k^2 - v\Delta_a(x)}\right) \frac{d}{dx}\psi(x). \quad (9)$$

Note that the R.H.S. of Eq. (9) is a product of two discontinuous quantities at $x = \pm a$, yielding the continuous quantity $\phi(x)$. Connection condition Eq. (7) can be rewritten in terms of $\psi(x)$ as

$$\psi(0_+) - \psi(0_-) = -\frac{v}{k^2}\psi'(0_+) = -\frac{v}{k^2}\psi'(0_-). \quad (10)$$

Differentiating Eq.(6), one has

$$-\frac{d^3}{dx^3}\phi(x) - v\frac{d}{dx}\Delta_a(x)\left(\frac{1}{k^2 - v\Delta_a(x)}\right)\phi(x) = k^2\frac{d}{dx}\phi(x). \quad (11)$$

We obtain an equation for $\psi(x)$ in the form

$$-\frac{d^2}{dx^2}\psi(x) - \left(\frac{v}{k^2 - v/(2a)}\right)\frac{d}{dx}\Delta_a(x)\frac{d}{dx}\psi(x) = k^2\psi(x). \quad (12)$$

We define

$$c = -\frac{v}{k^2}, \quad (13)$$

$$c_a = -\frac{v}{k^2 - v/(2a)}. \quad (14)$$

From these definitions, we have a relation

$$\frac{1}{c_a} = \frac{1}{c} + \frac{1}{2a}. \quad (15)$$

Rewriting Eq. (12), we now have a Schrödinger equation

$$-\frac{d^2}{dx^2}\psi(x) + c_a E_a(x)\psi(x) = k^2\psi(x) \quad (16)$$

with a potential

$$E_a(x) \equiv \frac{d}{dx}\Delta_a(x)\frac{d}{dx}. \quad (17)$$

One can see, from Eqs. (10) and (13), that the zero range limit $a \rightarrow 0$ of this potential results in the connection condition

$$\psi(0_+) - \psi(0_-) = c\psi'(0_+) = c\psi'(0_-), \quad (18)$$

when the coupling c_a is rescaled according to the relation, Eq.(15).

The operation of the E_a potential can be interpreted in two ways:

$$\langle \psi_2 | E_a | \psi_1 \rangle = \langle \psi_2 | \frac{d}{dx} \Delta_a \frac{d}{dx} \psi_1 \rangle = - \langle \psi_2' | \Delta_a | \psi_1' \rangle. \quad (19)$$

The second equation gives the expression

$$E_a(x) = - \frac{\vec{d}}{dx} \Delta_a(x) \frac{\vec{d}}{dx}. \quad (20)$$

This is essentially the form with which the discontinuity-inducing contact interaction has been first realized physically by Šeba [3].

III. THE EQUIVALENCE BETWEEN PRIME-DELTA-PRIME AND EPSILON FUNCTIONS

In this Section, we prove that the separable expression, which we have rederived in the last section, is identical to the local realization developed in Ref. [11]. We rewrite Eq. (11) in terms of $\psi(x)$ as

$$- \frac{d^2}{dx^2} \psi(x) + c_a \left(\frac{d}{dx} \Delta_a(x) \right) \frac{d}{dx} \psi(x) + c_a \Delta_a(x) \frac{d^2}{dx^2} \psi(x) = k^2 \psi(x). \quad (21)$$

The second term can be rewritten in two steps; first

$$\frac{d}{dx} \Delta_a(x) = \frac{1}{2a} (\delta(x+a) - \delta(x-a)), \quad (22)$$

and then

$$\begin{aligned} \frac{c_a}{2a} \delta(x+a) \frac{d}{dx} \psi(x) & \quad (23) \\ & \approx -c_a \cdot \frac{\delta(x+a_+) - \delta(x+0_+)}{2a^2} \psi(x) \\ \frac{c_a}{2a} \delta(x-a) \frac{d}{dx} \psi(x) & \\ & \approx c_a \cdot \frac{\delta(x-a_+) - \delta(x-0_+)}{2a^2} \psi(x). \end{aligned}$$

Here, a care must be taken to handle the quantity $\psi'(x)$ which is discontinuous at $x = \pm a$. The above argument is based on the convention that the quantities at $x = \pm a$ is evaluated as the limiting values from $|x| < a$ region, in accordance with Eq. (1). If one adopts the alternative definition of the step function such that $\Delta_a(\pm a) = 0$, one has to replace c_a in the second term of Eq. (21), and thus in L.H.S. of Eq. (23), by c (see Eqs. (9),(11)). However, the relations $c\psi'(a_+) = c_a\psi'(a_-)$ and $c\psi'(-a_+) = c_a\psi'(-a_-)$ guarantees that we still obtain the same result, R.H.S. of Eq. (23). Operations become easy after

this, since everything is expressed in terms of $\psi(x)$ which is continuous everywhere. We obtain

$$\begin{aligned} & - \{1 - c_a \Delta_a(x)\} \frac{d^2}{dx^2} \psi(x) \\ & - \frac{c_a}{2a^2} \{\delta(x+a_+) + \delta(x-a_+) - 2\delta(x)\} \psi(x) \\ & = k^2 \psi(x). \end{aligned} \quad (24)$$

We divide this by $(1 - c_a \Delta_a(x))$. Since this factor acts only within $|x| < a$, we have

$$\begin{aligned} & - \frac{d^2}{dx^2} \psi(x) - \frac{c_a}{2a^2} \{\delta(x+a) + \delta(x-a)\} \psi(x) \\ & + \frac{c_a/a^2}{1 - c_a \Delta_a(x)} \delta(x) \psi(x) \\ & = \frac{k^2}{1 - c_a \Delta_a(x)} \psi(x). \end{aligned} \quad (25)$$

This can be written in the form

$$- \frac{d^2}{dx^2} \psi(x) + \varepsilon_a(x; c) \psi(x) = k^2 \psi(x), \quad (26)$$

where

$$\begin{aligned} \varepsilon_a(x; c) & = - \frac{c_a}{2a^2} [\delta(x+a) + \delta(x-a)] \\ & + \frac{c_a}{a^2} \cdot \frac{1}{1 - c_a/(2a)} \delta(x) \\ & - \frac{c_a}{2a} \cdot \frac{k^2}{1 - c_a/(2a)} \Delta_a(x) \end{aligned} \quad (27)$$

represents the equivalent local potential of $E_a(x)$. Using the expressions

$$\begin{aligned} \frac{c_a}{2a} & \approx 1 - \frac{2a}{c} + O(a^2), \\ \frac{c_a}{1 - c_a/(2a)} & = c, \end{aligned} \quad (28)$$

one arrives at

$$\varepsilon_a(x; c) = \left(\frac{2}{c} - \frac{1}{a} \right) \{\delta(x+a) + \delta(x-a)\} + \frac{c}{a^2} \delta(x). \quad (29)$$

This is exactly the ε -function, a local expression given in Ref. [11] for the discontinuity-inducing interaction in one-dimensional quantum mechanics.

IV. PERTURBATIVE RENORMALIZABILITY UP TO SECOND ORDER

Despite its intuitive nature and usefulness in “engineering” purpose, the ε -function expression has a major setback in its apparent inability to cope with perturbative approach. One realizes this fact easily by calculating the

matrix element of $\varepsilon_a(x; c)$. For one thing, the coupling constant c appears in its inverse. The limit $a \rightarrow 0$ immediately gives the divergence. The separable expression $c_a E_a(x)$ looks promising in that respect. In this section, we show that the perturbation in terms of c_a along with the renormalization procedure indeed gives the sensible answer through the explicit calculation of energy eigenvalue up to second order.

Consider

$$-\frac{d^2}{dx^2}\psi(x) + V(x)\psi(x) = k^2\psi(x) \quad (30)$$

with $V(x)$ being either $v\Delta_a(x)$ or $c_a E_a(x)$ with disappearing a . For definiteness, we place the system on the line of length L with periodic boundary

$$\psi(-L/2) = \psi(L/2), \quad \psi'(-L/2) = \psi'(L/2). \quad (31)$$

The problem now becomes that of free wave equation on a “ring” with a defect at $x = 0$ whose characteristics is specified by the zero-range function $V(x)$. The solution which satisfies the condition Eq. (31) takes the form

$$\begin{aligned} \psi(x) &= A \sin k(x - \eta) & (x > 0) \\ &A \sin k(x - \eta + L) & (x < 0), \end{aligned} \quad (32)$$

where A and η are the constants which is to be determined. For $V = v\Delta_a(x)$ with $a \rightarrow 0$, the connection condition Eq. (7) gives

$$\tan \frac{kL}{2} = \frac{v}{2k}. \quad (33)$$

From this equation, the expansion of k^2 in terms of the free particle solution κ define by

$$-\frac{d^2}{dx^2}\varphi(x) = \kappa^2\varphi(x) \quad (34)$$

is easily obtained as

$$k^2 \approx \kappa^2 + \frac{2}{L}v - \frac{1}{L^2\kappa^2}v^2 + \dots \quad (35)$$

Similarly, for $V(x) = c_a E_a(x)$ at $a \rightarrow 0$, the connection condition Eq. (17) gives

$$\tan \frac{kL}{2} = -\frac{kc}{2}, \quad (36)$$

which yields

$$k^2 \approx \kappa^2 - \frac{2\kappa^2}{L}c + \frac{3\kappa^2}{L^2}c^2 + \dots \quad (37)$$

Suppose we are interested in obtaining these results through perturbation theory that starts from the free problem, Eq. (34) whose solution is given by

$$\begin{aligned} \varphi_n(x) &= \frac{1}{\sqrt{L}}e^{i\kappa_n x}, \\ \kappa_n &= \frac{2\pi}{L}n \quad (n = 0, \pm 1, \pm 2, \dots). \end{aligned} \quad (38)$$

The Rayleigh-Schrödinger formula reads

$$k_n^2 \approx \kappa_n^2 + \langle n|V|n \rangle + \sum'_m \frac{\langle n|V|m \rangle \langle m|V|n \rangle}{\kappa_n^2 - \kappa_m^2} \dots, \quad (39)$$

where \sum' represents the exclusion of $m = n$, and the matrix elements are given by

$$\langle m|V|n \rangle \equiv \int_{-\infty}^{\infty} dx \varphi_m^*(x)V(x)\varphi_n(x). \quad (40)$$

Because of the degeneracy $\kappa_{-n} = \kappa_n$, it is advantageous to use the symmetrized and antisymmetrized wave functions.

$$\varphi_{n\pm} \equiv \frac{1}{\sqrt{1 + \delta_{n0}}} \cdot \frac{1}{\sqrt{2}}(\varphi_n \pm \varphi_{-n}). \quad (41)$$

Because of the mirror symmetry $V(-x) = V(x)$, one has

$$\langle m_{\pm}|V|n_{\mp} \rangle = 0 \quad (42)$$

which guarantees the separation of the state vectors into the $\{n_+\}$ and $\{n_-\}$ sectors. within each sector, one has

$$\begin{aligned} \langle m_{\pm}|V|n_{\pm} \rangle &= \frac{1}{\sqrt{(1 + \delta_{n0})(1 + \delta_{m0})}} \\ &\times \{ \langle m|V|n \rangle \pm \langle m|V|-n \rangle \}. \end{aligned} \quad (43)$$

Explicit forms of the matrix elements, Eq. (40) are given by

$$\begin{aligned} \langle m_{\pm}|\Delta_a|n_{\pm} \rangle &= \frac{1}{\sqrt{(1 + \delta_{n0})(1 + \delta_{m0})}} \cdot \frac{1}{L} \\ &\times \left(\frac{\sin a\kappa_{nm}^-}{a\kappa_{nm}^-} \pm \frac{\sin a\kappa_{nm}^+}{a\kappa_{nm}^+} \right), \end{aligned} \quad (44)$$

$$\begin{aligned} \langle m_{\pm}|E_a|n_{\pm} \rangle &= \frac{-1}{\sqrt{(1 + \delta_{n0})(1 + \delta_{m0})}} \cdot \frac{\kappa_m \kappa_n}{L} \\ &\times \left(\frac{\sin a\kappa_{nm}^-}{a\kappa_{nm}^-} \mp \frac{\sin a\kappa_{nm}^+}{a\kappa_{nm}^+} \right), \end{aligned} \quad (45)$$

where the notation

$$\kappa_{nm}^{\pm} \equiv \kappa_n \pm \kappa_m \quad (46)$$

is adopted, and it is understood that for $m = n$, $(\sin a\kappa_{mn}^-)/(a\kappa_{mn}^-)$ is set to be 1. From these expressions, it is easy to see that at $a \rightarrow 0$ limit, we have

$$\langle m_-|\Delta_a|n_- \rangle \rightarrow 0, \quad (47)$$

$$\langle m_+|E_a|n_+ \rangle \rightarrow 0. \quad (48)$$

Therefore, it is sufficient to consider only $\{n_+\}$ for Δ_a and $\{n_-\}$ for E_a . Note the fact that for $\{n_-\}$, only $n \geq 1$ is allowed, while for $\{n_+\}$, n can be any non-negative value including 0.

We start with the case of Δ_a . Assuming the unperturbed state is of positive energy $\kappa_n > 0$ (therefore $n \neq 0$), one has

$$\langle n_+ | \Delta_a | n_+ \rangle = \frac{2}{L}, \quad (49)$$

and

$$\begin{aligned} & \sum_m \prime \frac{|\langle n_+ | \Delta_a | m_+ \rangle|^2}{\kappa_n^2 - \kappa_m^2} \\ &= \frac{1}{L^2} \sum_{m=0}^{\infty} \prime \frac{1}{\kappa_n^2 - \kappa_m^2} \left(\frac{\sin a\kappa_{nm}^-}{a\kappa_{nm}^-} + \frac{\sin a\kappa_{nm}^+}{a\kappa_{nm}^+} \right)^2 \frac{1}{1 + \delta_{m0}} \\ &\approx \frac{1}{L^2} \left[\sum_{m=1}^{\infty} \prime \frac{4}{\kappa_n^2 - \kappa_m^2} + \frac{2}{\kappa_n^2} \right] \\ &= -\frac{1}{L^2 \kappa_n^2}, \end{aligned} \quad (50)$$

where the limit $a \rightarrow 0$ is taken at the second line, and a well known summation formula is used in the last line. Thus one recovers the correct expansion, Eq. (34), which is of course a well-known result from elementary textbooks.

Now we consider the perturbation by E_a . The first order term is easily obtained as

$$\langle n_- | E_a | n_- \rangle = -\frac{2\kappa_n^2}{L}, \quad (51)$$

The second order term is given by

$$\begin{aligned} & \sum_m \prime \frac{|\langle n_- | E_a | m_- \rangle|^2}{\kappa_n^2 - \kappa_m^2} \\ &= \frac{1}{L^2} \sum_{m=1}^{\infty} \prime \frac{\kappa_n^2 \kappa_m^2}{\kappa_n^2 - \kappa_m^2} \left(\frac{\sin a\kappa_{nm}^-}{a\kappa_{nm}^-} + \frac{\sin a\kappa_{nm}^+}{a\kappa_{nm}^+} \right)^2 \\ &= \frac{1}{L^2} \left[\kappa_n^4 \sum_{m=1}^{\infty} \prime \frac{1}{\kappa_n^2 - \kappa_m^2} \left(\frac{\sin a\kappa_{nm}^-}{a\kappa_{nm}^-} + \frac{\sin a\kappa_{nm}^+}{a\kappa_{nm}^+} \right)^2 \right. \\ & \quad \left. - \kappa_n^2 \sum_{m=1}^{\infty} \prime \left(\frac{\sin a\kappa_{nm}^-}{a\kappa_{nm}^-} + \frac{\sin a\kappa_{nm}^+}{a\kappa_{nm}^+} \right)^2 \right]. \end{aligned} \quad (52)$$

In the second equation, the first term can be readily calculated by taking the $a \rightarrow 0$ limit as in the case of Δ_a :

$$\begin{aligned} & \sum_{m=1}^{\infty} \prime \frac{1}{\kappa_n^2 - \kappa_m^2} \left(\frac{\sin a\kappa_{nm}^-}{a\kappa_{nm}^-} + \frac{\sin a\kappa_{nm}^+}{a\kappa_{nm}^+} \right)^2 \\ &\approx \sum_{m=1}^{\infty} \prime \frac{4}{\kappa_n^2 - \kappa_m^2} \\ &= -\frac{3}{\kappa_n^2}. \end{aligned} \quad (53)$$

The divergence of the second term at $a \rightarrow 0$ limit has to be explicitly handled. Using the relation in the Appendix, one obtains

$$\begin{aligned} \sum_{m=1}^{\infty} \prime \left(\frac{\sin a\kappa_{nm}^-}{a\kappa_{nm}^-} + \frac{\sin a\kappa_{nm}^+}{a\kappa_{nm}^+} \right)^2 &\approx \frac{2\pi}{2\pi a/L} - 6 \\ &= \frac{L}{a} - 6. \end{aligned} \quad (54)$$

We have

$$\sum_m \prime \frac{|\langle n_- | E_a | m_- \rangle|^2}{\kappa_n^2 - \kappa_m^2} \approx 3\frac{\kappa_n^2}{L^2} - \frac{\kappa_n^2 L}{a}. \quad (55)$$

Therefore, the summation to the second order, Eq. (39) gives

$$k^2 \approx \kappa^2 \left\{ 1 - \frac{2}{L}c_a + \frac{3}{L^2}c_a^2 - \frac{1}{La}c_a^2 \right\}. \quad (56)$$

The appearance of the divergence calls for the renormalization procedure. The replacement of bare coupling c_a by the renormalized coupling c serves the purpose. Using the expansion of Eq. (18) in terms of $1/a$

$$c_a = \frac{2ac}{2a+c} \approx c - \frac{c^2}{2a} + \dots, \quad (57)$$

we obtain

$$k^2 \approx \kappa^2 \left\{ 1 - \frac{2}{L}c + \frac{1}{La}c^2 + \frac{3}{L^2}c^2 - \frac{1}{La}c^2 \right\}. \quad (58)$$

Thus the divergence is cancelled out, and the correct result Eq. (36) is reproduced.

It should be possible to go on to higher order of perturbation and to show the cancellation of divergent term order by order. Also, perturbative calculation of wave function could be similarly performed. But the calculation up to the second order shown here would be sufficient to convince the validity of perturbative renormalization in this model.

An interesting fact of the perturbative expansion for $c_a E_a(x)$, Eq. (37) is that it is formally identical to the strong coupling expansion of $v\Delta_a(x)$ in terms of $1/v$. This, of course, is not an accident but the result of the duality between δ and ε potentials [8,13].

V. CONCLUSION

Now that a calculational scheme is devised to deal with the divergences arising in the perturbative calculation of discontinuity-inducing interaction, we stand at the starting line to tackle more practical physical problems. Simplest among them is the system that has both regular finite-range potential *and* the contact force. More interesting is the finite range potential problem that inherently requires the consideration of the singular zero-range

force, that of $1/|x|$, or “one-dimensional Coulomb” problem as is sometimes called [14,15]. Our approach would be profitably applied also to the many-body problem of one-dimensional particles where the second-quantized representation is frequently utilized. Another direction for the potential development is the generalization of our analysis to the relativistic quantum mechanical models, and ultimately to the field theoretical models. Such analysis might be useful to shed some light on the relationship between the fermion-boson dualities found in quantum mechanics and in field theories [8,16]. It is known that one can construct three parameter family of generalized contact interaction by combining the δ and ε interactions [11]. A detailed study of most general contact interaction in one dimensional quantum mechanics and its possible relativistic and field theoretical extension should be also of great interest.

Prior to the proper formulation of the discontinuity-inducing contact force, there has been an introduction of a set of interactions known in nuclear physics as Skyrme force [17] which includes a component analogous to prime-delta-prime force, Eq. (20) in *spatial dimension three*. With such interactions, a rather detailed numerical analysis has been carried out [18]. However, from the current view, it is clear that the results drawn from such analysis are in need of critical reexamination, since such object cannot be defined as renormalizable interaction in other dimensions than one. This is to be contrasted to the δ interaction which *can* be constructed as renormalizable interactions in dimensions two and three [4].

Throughout our treatment, we have had no need to invoke any of the high mathematics traditionally associated with the theory of generalized contact interactions. In fact, it has been felt that the current analysis might be useful as pedagogical materials for the graduate level textbook on quantum mechanics. Specifically, the great simplification garnered by the solvability of problems helps detach the concept of the renormalization from the complex machinery of field theory. It enables one to readily recognize that the two facets of renormalization, namely, the invariance of the problem with suitable rescaling, and a technical procedure that allows the perturbative treatment, are one and the same thing.

Finally we would like to reiterate our basic message that the discontinuity-inducing interaction, a lesser known twin of δ function, now calls for a proper recognition as an integral element of the quantum mechanics.

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APPENDIX

We outline the proof of the summation formula

$$\begin{aligned} & \frac{1}{\beta^2} \sum_{m=1}^{\infty} \left(\frac{\sin \beta(n-m)}{n-m} + \frac{\sin \beta(n+m)}{n+m} \right)^2 \quad (59) \\ &= \frac{2\pi}{\beta} - 6 + O(\beta) \end{aligned}$$

in the limit $\beta \rightarrow 0$. It can be split into two steps: First, we have a well known relation

$$\sum_{m=1}^{\infty} \frac{\sin^2 \beta m}{m^2} = \frac{1}{2} \beta (\pi - \beta). \quad (60)$$

Next we can show, for small β ,

$$\begin{aligned} & \frac{1}{\beta^2} \sum_{m=1}^{\infty} \left(\frac{\sin \beta(n-m)}{n-m} + \frac{\sin \beta(n+m)}{n+m} \right)^2 \quad (61) \\ &+ 4 - \frac{4}{\beta^2} \sum_{m=1}^{\infty} \frac{\sin^2 \beta m}{m^2} \\ &\approx \frac{1}{\beta} \int_0^{\infty} dx \left[\left(\frac{\sin(\beta n - x)}{\beta n - x} + \frac{\sin(\beta n + x)}{\beta n + x} \right)^2 \right. \\ &\quad \left. - \frac{4 \sin^2 x}{x^2} \right] \\ &+ \left[4 - \left(1 + \frac{\sin 2\beta n}{2\beta n} \right)^2 \right] \\ &= O(\beta). \end{aligned}$$

It is easy to see that one obtains Eq. (59) by combining these two equations.

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