Quantum Abacus: 1D Locational Qubit Implementation

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Abstract. It is shown that the $U(2)$ group structure of one-dimensional contact interaction working as a barrier can be adopted for quantum information processing when used in combination with environmental potential whose bouncing modes are profile preserving. Qubits are realized as wave functions localized in either side of the barrier which divides the system into two regions. It is argued that this model is a theoretical prototype of a robust and scalable quantum computing device.

Keywords: quantum computation, quantum contact interaction, quantum wire

1 Introduction

Quantum information processing is based on quantum states belonging to $U(2)$ group. Typically, spin one-half is considered as a natural playground. But any two level system can be utilized. An intriguing $U(2)$ structure has been uncovered in one dimensional system with generalized point interaction [1, 2, 3]. A natural question is whether it can be used for quantum information processing.

The purpose of this talk is to show that this is indeed possible with the help of background potential, which move the particle back and forth while keeping the wave function profile intact. We consider controlling that motion by manipulating the properties of a barrier which separates the two spacial regions of the system. The quantum barrier, when it is sufficiently thin, can be modeled by the point interaction whose characteristics is specified by $U(2)$. When the wave functions localized at each of the two separated regions are identified as qubits $|0\rangle$ and $|1\rangle$, it can be shown that this parametric $U(2)$ induces the corresponding $U(2)$ operation on qubit states.

The resulting model system takes the appearance of the quantum version of the ancient eastern calculational device of *abacus*. We argue that this model could be a prototype of a robust qubit device which holds advantage in stability, controllability and scalability.

2 The Model

We consider a one dimensional system of quantum particle moving on x axis subjected to a harmonic oscillator potential of frequency $\omega = 2\pi/T$ and the inverse square potential with the strength g;

$$V(x) = \frac{\omega^2}{2}x^2 + g\frac{1}{x^2}. \qquad (1)$$

With $g = 0$, the background potential is reduced to the elementary harmonic oscillator. In this article, we shall mainly work with this simple limit. Then, we further add the generalized point interaction placed at the origin whose characteristics are controllable. We define boundary vector at $x \to +0$ and $x \to -0$ as

$$\Psi = \begin{pmatrix} \psi(0_+) \\ \psi(0_-) \end{pmatrix}, \quad \Psi' = \begin{pmatrix} \psi'(0_+) \\ -\psi'(0_-) \end{pmatrix}. \qquad (2)$$

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The point interaction is described by an element U of unitary group $U(2)$, that specifies the value of the boundary vector such that

$$(U - I)\Psi + i(U + I)\Psi' = 0. \qquad (3)$$

In other word, all point interactions allowable in quantum mechanics form a family described by the set of four parameters, whose manifold structure is given by $U(2) \simeq S^1 \times S^3$ [4].

The elementary example of δ-interaction is but a very special one parameter family within this wider class of interactions. In fact, generalized point interaction $U(2)$ comprises such exotic interactions that cause discontinuity in the wave function itself, and also the ones that have constant transmission probability which is independent of the particle energy.

Explicit construction of these highly singular point interactions has been achieved in terms of singular short-range limits of known interactions [5]. Here, we only illustrate some of the prominent examples. The identity matrix $U = I$ results in $\psi'(0_+) = \psi'(0_-) = 0$, signifying the inpenetrable barrier at $x = 0$ with Neumann boundary at its both side. Similarly, the negative ofidentity matrix $U = -I$ results in $\psi(0_+) = \psi(0_-) = 0$, another inpenetrable wall with Dirichlet boundary. Curiously, it is $U = \sigma_1$ that results in barrier-less free oscillater, in $\psi(0_+) = \psi(0_-)$, $\psi'(0_+) = \psi'(0_-)$. The case of $g \neq 0$ is somewhat harder for the intuition, but analogous interpretation can be made. For example, with $U = \pm I$ the two sides $x > 0$ and $x < 0$ are non-communicating, while $U = \sigma_3$ represents maximal communication. An inportant example is the case of Hadamard matrix $U = H$ $= (\sigma_1 + \sigma_3)/\sqrt{2}$. It turns out that this point interaction lets the particle with all energy transmit by probability one-half.

Our central assertion is that the parametric $U(2)$ describing the whole family of quantum point interaction can be turned into a *Hilbert space* $U(2)$ *of particle states* in the following manner: If we identify the state localized at one side (let's say, right side) as qubit $|0\rangle$ and the other side as $|1\rangle$, after half-period $T/2$, arbitrary state $|\psi\rangle = \alpha_0 |0\rangle + \alpha_1 |1\rangle$ is turrned into $U |\psi\rangle$.

Essential first step for the proof is the decomposition of $S^1 \times S^3$ into the spectral torus T^2 and the isospectral sphere S^2 [4]. The decomposition is realized, in terms of an element $U \in U(2)$ as

$$U = \sigma D \sigma \tag{4}$$

where

$$D = \begin{pmatrix} e^{i\theta_+} & 0 \\ 0 & e^{i\theta_-} \end{pmatrix}, \quad \sigma = \begin{pmatrix} \cos\frac{\mu}{2} & e^{i\nu}\sin\frac{\mu}{2} \\ e^{-i\nu}\sin\frac{\mu}{2} & -\cos\frac{\mu}{2} \end{pmatrix}. \tag{5}$$

The range of the parameters are given by $0 \leqslant \theta_\pm < 2\pi$, $0 \leqslant \mu < \pi$ and $0 \leqslant \nu < 2\pi$. The parameter space $\{\theta_+, \theta-\}$ forms a torus and $\{\mu, \nu\}$ a sphere. The latter is to be indentified with the Bloch sphere in quantum computation terminology. A noteworthy feature of the system is that the energy levels are decided solely D and changing σ will keep the spectra.

An important observation is that any element D belonging to the torus $\{\theta_+, \theta-\}$ can be expressed as a product of two Bloch elements and an overall phase $D = e^{i\xi}\,\sigma_a\,\sigma_b$ with suitably chosen σ_a and σ_b [4]. Since the phase $e^{i\xi}$ may be cancelled out by adding a constant potential to the Hamiltonian (1), we find that any unitary operation U can be performed essentially by the successive application of σ belonging to the sphere $\{\mu, \nu\}$ in up to four steps.

3 Bloch Sphere Imprinting

We now look at the the effect of the operation of σ on wave functions. The connection conditions of wave function at the origin is given by

$$\psi(0_+) = \lambda\psi(0_-), \quad \psi'(0_+) = \frac{1}{\lambda^*}\psi'(0_-), \tag{6}$$

$$\text{where} \quad \lambda = e^{i\nu}\sqrt{\frac{1+\cos\frac{\mu}{2}}{1-\cos\frac{\mu}{2}}}.$$

The eigenvalues of the system are unchanged from the free harmonic oscillator's $\{n\omega\}$, and its eigenfunctions of the system $\{\chi_n^\lambda\}$ are also analoguous to the free one $\{\chi_n\}$ having discontinuity at the origin and being expanded/shrank at one side;

$$\begin{aligned} \chi_n^\lambda(x) &= N\left[\lambda\chi_n(|x|)\Theta(x) + \chi_n(|x|)\Theta(-x)\right] \\ & \qquad n = 0, 2, 4, \cdots, \\ \chi_n^\lambda(x) &= N\left[\chi_n(|x|)\Theta(x) - \lambda^*\chi_n(|x|)\Theta(-x)\right] \\ & \qquad n = 1, 3, 5, \cdots, \end{aligned} \tag{7}$$

where the normalization is given by $N = \sqrt{2/(|\lambda|^2+1)}$. Arbitrary state $\psi(x,t)$ is represented as

$$\begin{aligned} \psi(x,t) &= \sum_n A_n \chi_n^\lambda(x) e^{in\omega t} \\ &= \frac{\lambda}{|\lambda|^2+1}\left[\lambda^* S(x) - M(x)e^{i\omega t}\right]\Theta(x)e^{i2m\omega t} \\ &+ \frac{\lambda^*}{|\lambda|^2+1}\left[S(x) + \lambda M(x)e^{i\omega t}\right]\Theta(-x)e^{i2m\omega t} \end{aligned} \tag{8}$$

where we define

$$\begin{aligned} S(x) &= \frac{2}{\lambda^* N}\sum_m A_{2m}\chi_{2m}(|x|), \\ M(x) &= -\frac{2}{\lambda N}\sum_m A_{2m+1}\chi_{2m+1}(|x|). \end{aligned} \tag{9}$$

Let's assume that at $t = 0$, the wave function is localized in the region $x > 0$. That is possible only if we have $M(x) = -S(x)/\lambda$. We then have

$$\begin{aligned} \psi(x,0) &= \Theta(x)\ S(x) \\ \psi(x,\frac{T}{2}) &= \left[\cos\frac{\mu}{2}\Theta(x) + e^{i\nu}\sin\frac{\mu}{2}\Theta(-x)\right] S(x). \end{aligned} \tag{10}$$

Similarly, with $S(x) = M(x)/\lambda^*$, we have

$$\begin{aligned} \psi(x,0) &= \Theta(-x)\ M(x) \\ \psi(x,\frac{T}{2}) &= \left[e^{-i\nu}\sin\frac{\mu}{2}\Theta(x) - \cos\frac{\mu}{2}\Theta(-x)\right] M(x). \end{aligned} \tag{11}$$

Thus, our assertion is proven for arbitrary Bloch sphere element σ.

Note that the present realization of unitary operations associated with the Bloch sphere is available for an arbitrary state, not just for a particular pair of eigenstates. This allows one to consider a qubit space spanned by a state with an *arbitrary* profile localized on one side of the barrier and its mirror state. This guarantees the smooth and robust transition to the semiclassical regime, although the problem of the coherence is still to be solved.

4 Prospects

In mathematical term, the two-qubit operation comprises $U(4)$ group. This is a natural extension to the $U(2)$, whose quantum wire realization has been the subject of this work. Analogous realization of this $U(4)$ exists in the form of "quantum X-junction", a graph of four half lines whose edges are connected at single point. A practical question in the experimental realization of our scheme is how to change the characteristics of the thin barrier representing the generalized point interaction. Preferably, it is to be a quantally operating device with triggering mechanism utilizing particles of far smaller mass or energy scale compared to the particle used as the qubit carrier. Once such device is constructed (no doubt that will be done, in time), and if that trigger is coupled to the presence absence of the qubit carrier in neighboring device, we will have a realization of two-qubit operations such as control-not in place.

The content of the current work is reported elsewhere in full [6].

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