Dynamics and Statistics of Quantum Turbulence in Quantum Fluid

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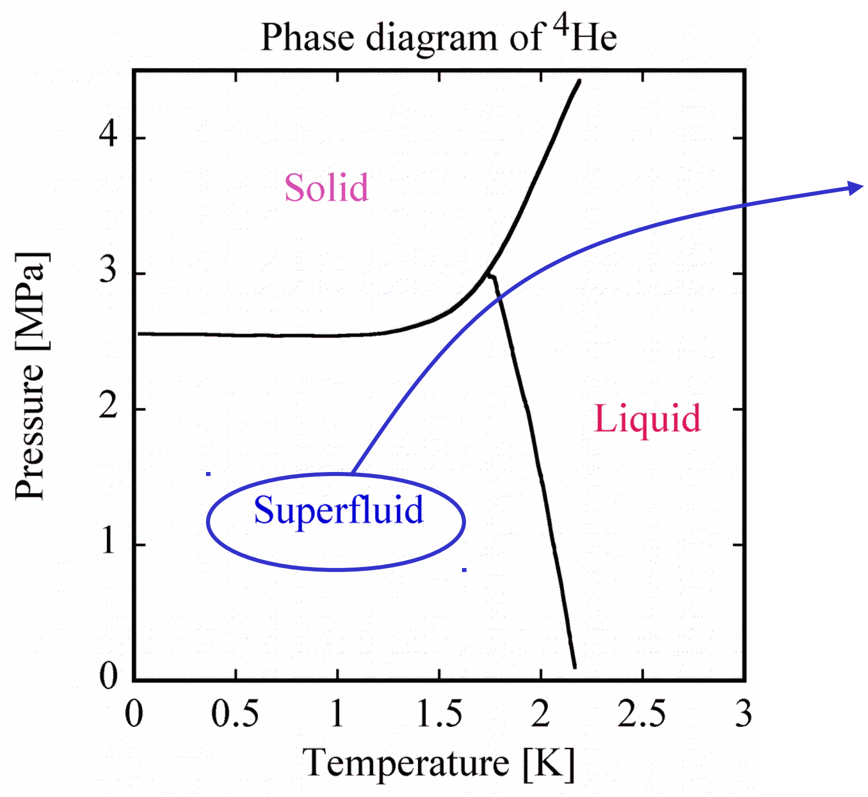
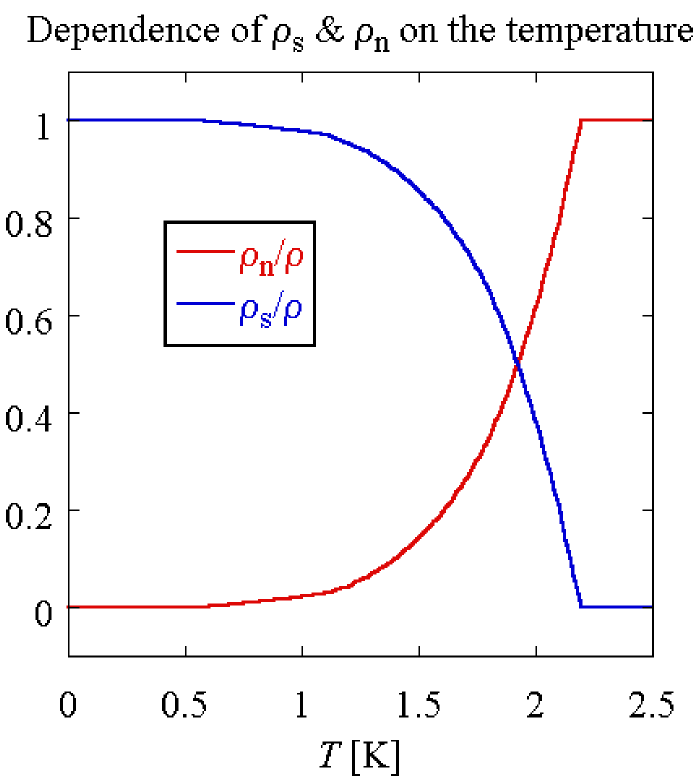
May 25, 2006, Kansai Seminar House

Contents

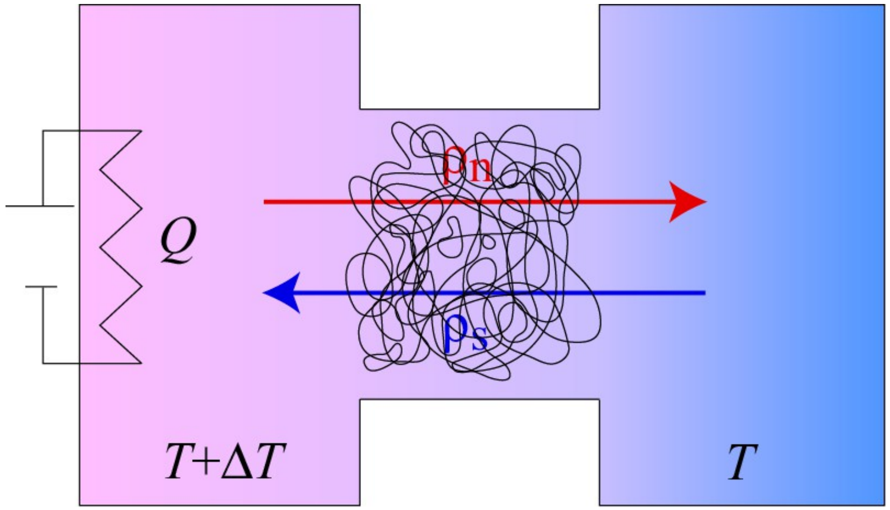
1, Introduction -History of Quantum Turbulence-.





Thermal Counter Flow and Superfluid Turbulence

Thermal counter flow in the temperature gradient

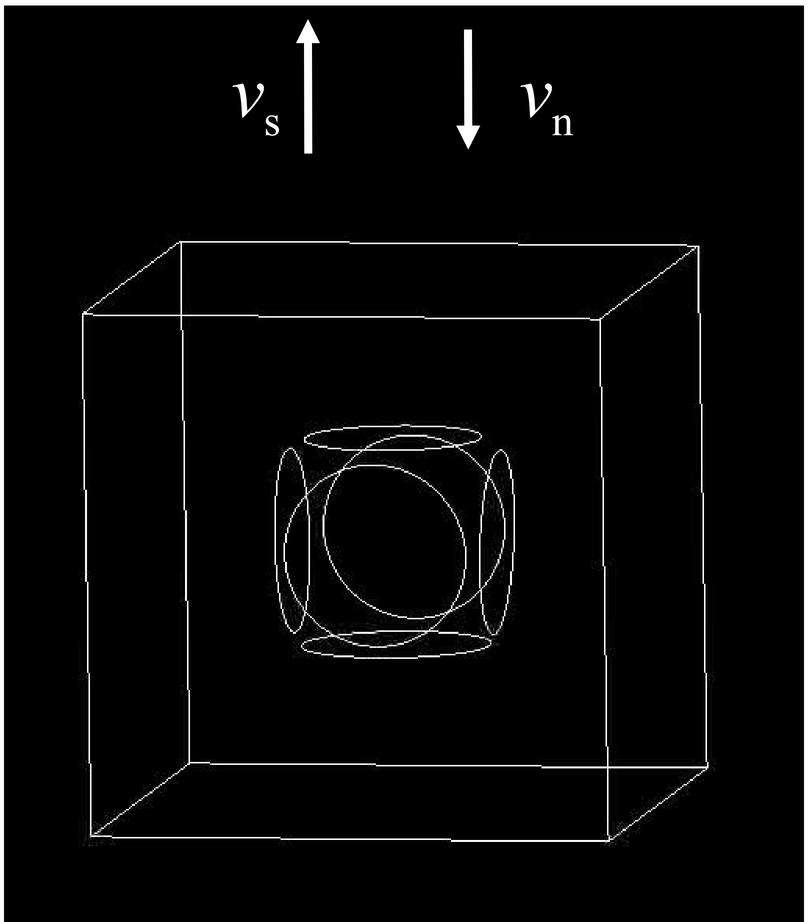


Above a critical velocity

Superfluid Turbulence is realized in the thermal counter flow (By Vinen, 1957)

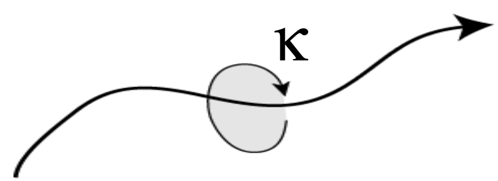
Superfluid Turbulence : Tangled State of Quantum Vortices

Vortex tangle in superfluid turbulence

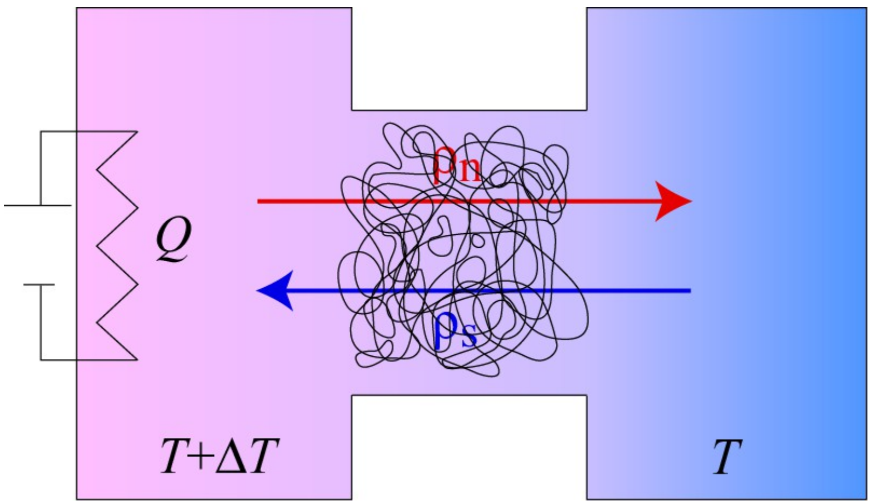


Quantized Vortex

- All Vortices have a same circulation $\kappa = \oint \boldsymbol{v}_s \cdot d\boldsymbol{s} = h / m$.
- Vortices can be stable as topological defects (not dissipated).
- Vortices have very thin cores (~Å for ^{4}He) : Vortex filament model is realistic



What Is The Relation Between Classical and Superfluid Turbulence?



Thermal counter flow had been main method to create superfluid turbulence until 1990's

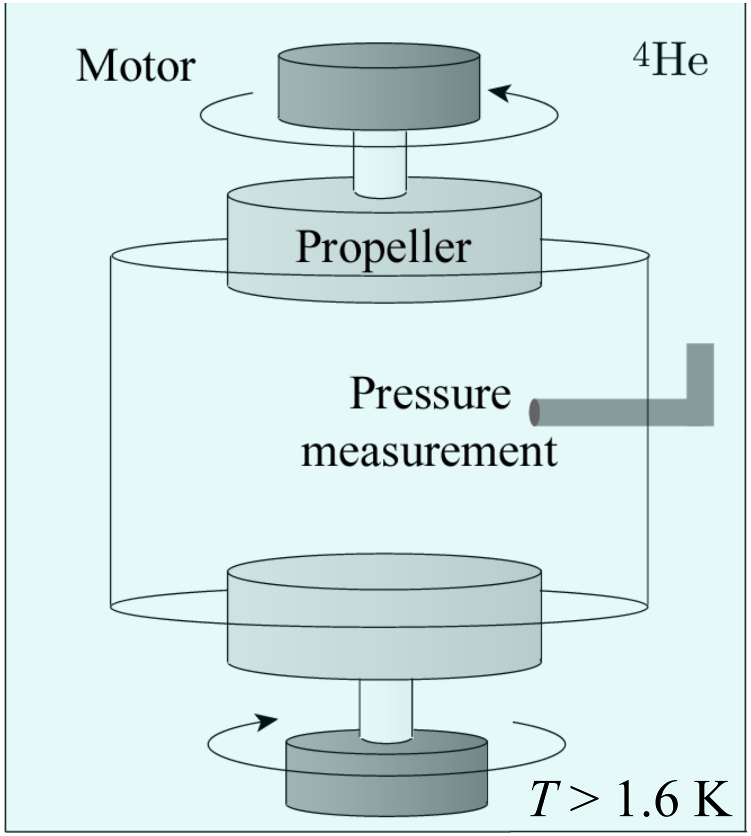
↓

Thermal counter flow has no analogy with classical fluid dynamics

The relation between superfluid and classical turbulence had been one great mystery.

Opening a New Stage in the Study of Superfluid Turbulence

J. Maurer and P. Tabeling, Europhys. Lett. **43** (1), 29 (1998)

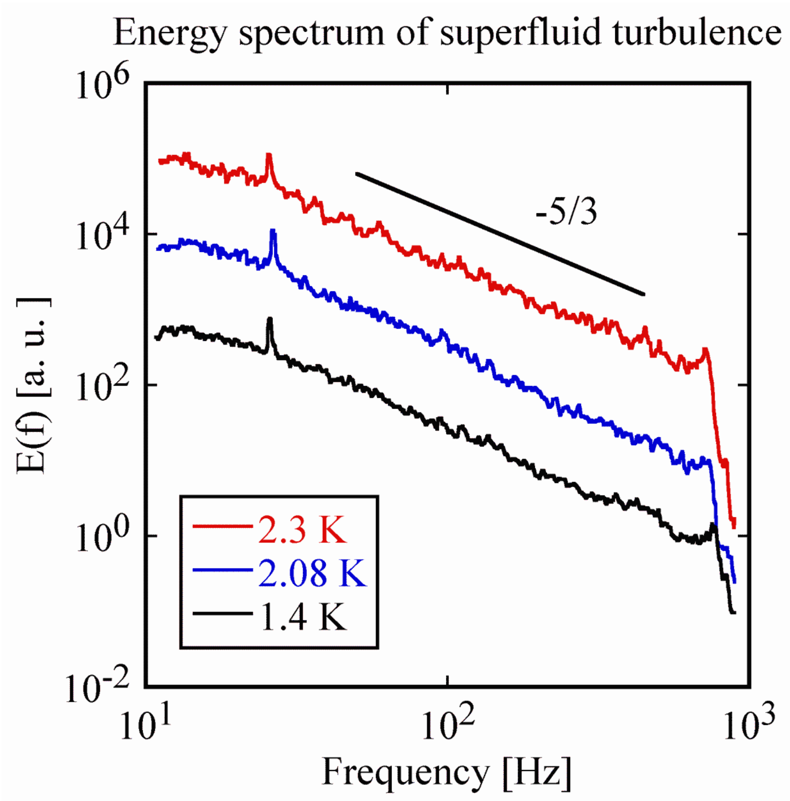


Two-counter rotating disks

Similar method to create classical turbulence : It becomes possible to discuss the relation between superfluid and classical turbulence

Energy Spectrum of Superfluid Turbulence

J. Maurer and P. Tabeling, Europhys. Lett. **43** (1), 29 (1998)

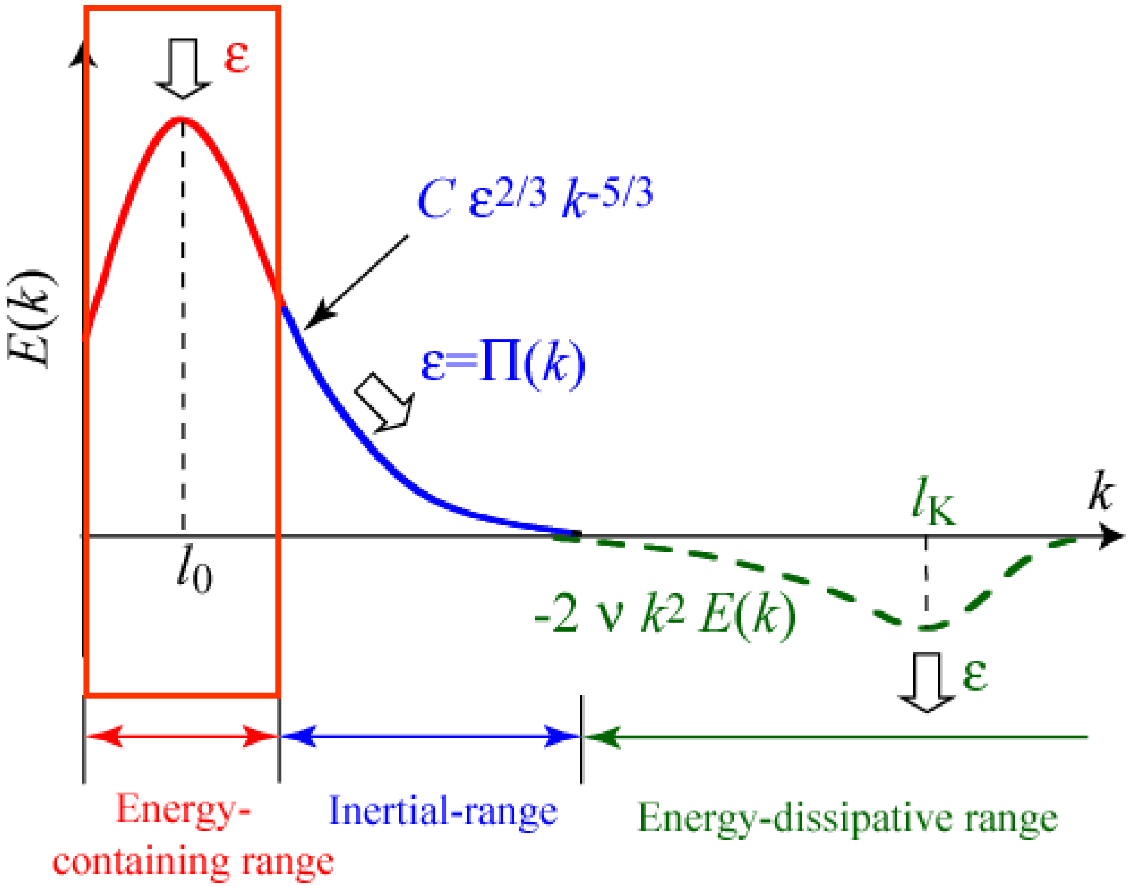


Even below the superfluid critical temperature, Kolmogorov –5/3 law was observed.

Similarity between superfluid and classical turbulence was obtained!

Kolmogorov Law : Statistical Law of Classical Turbulence

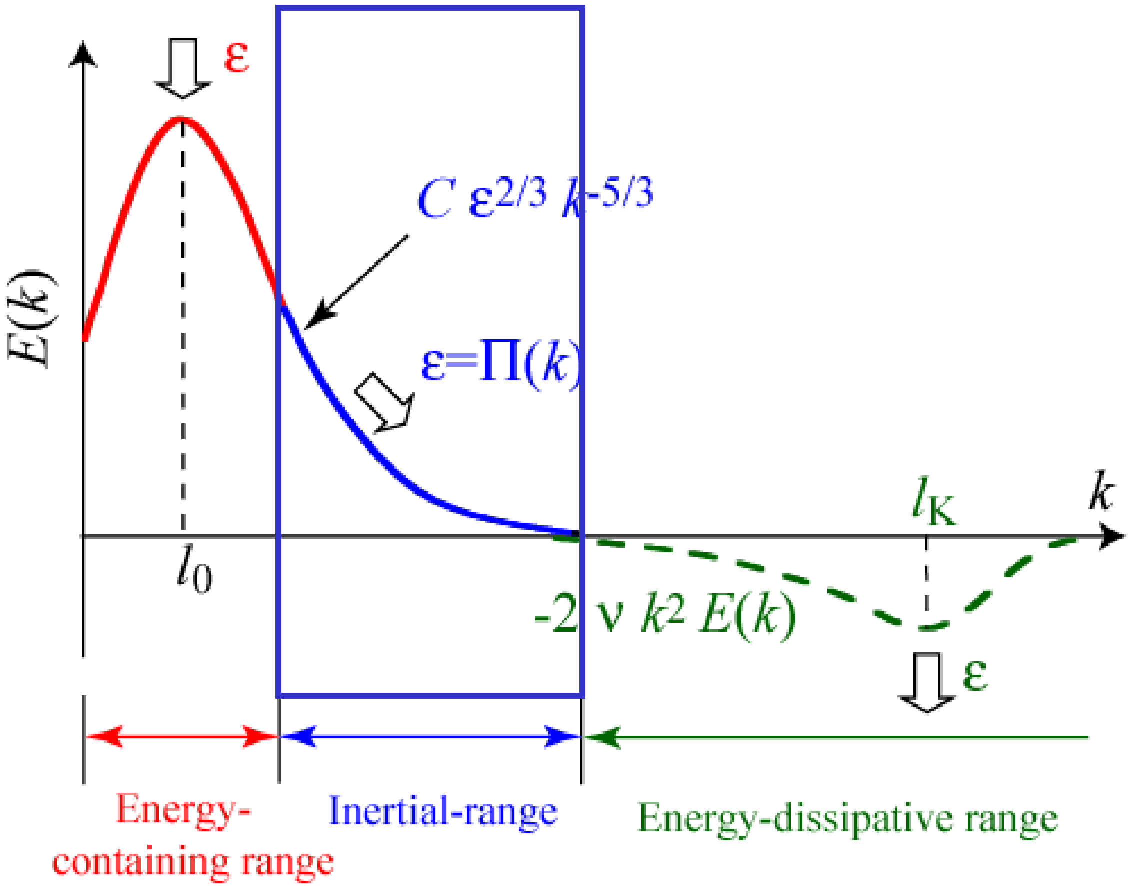
Homogeneous, isotropic, incompressible and steady turbulence



In the energy-containing range, energy is injected to system at scale l_0

Kolmogorov Law : Statistical Law of Classical Turbulence

Homogeneous, isotropic, incompressible and steady turbulence



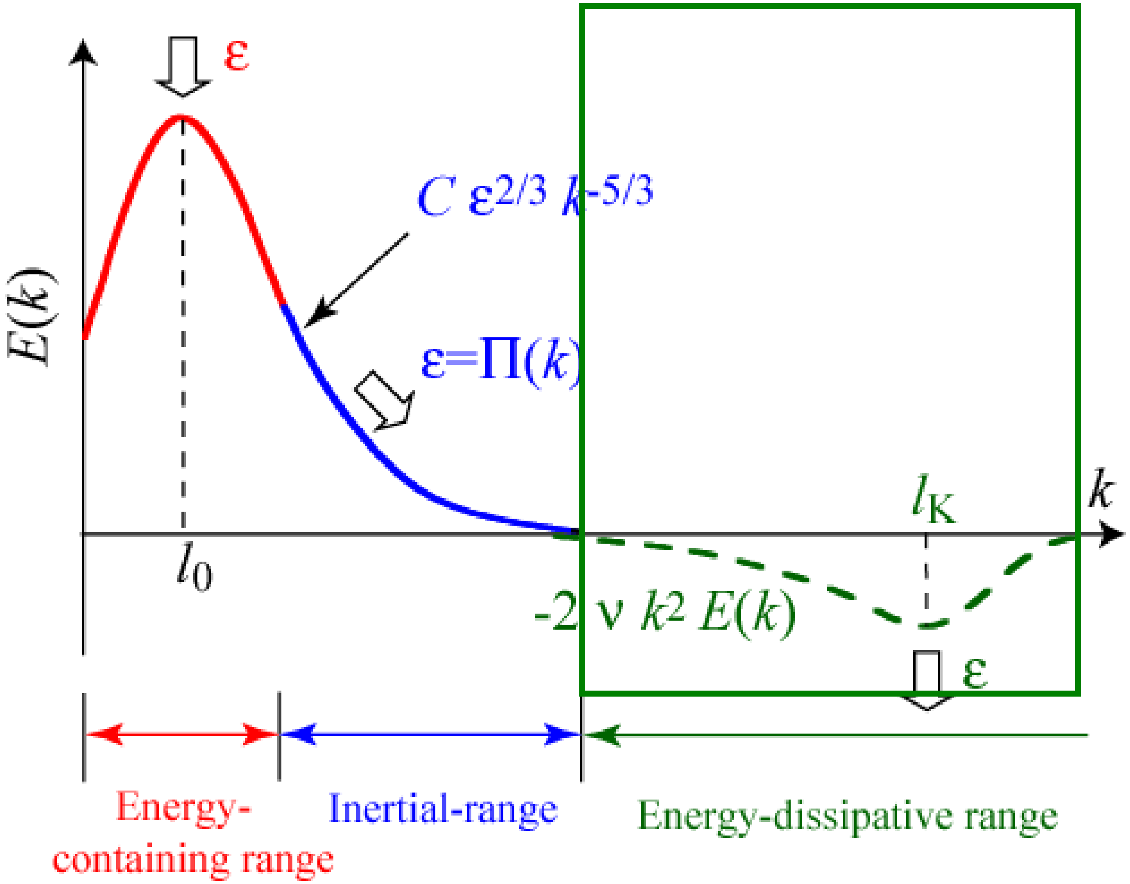
In the inertial range, the scale of energy becomes small without being dissipated, supporting Kolmogorov energy spectrum $E(k)$.

$$E(k) = C\varepsilon^{2/3}k^{-5/3}$$

C : Kolmogorov constant

Kolmogorov Law : Statistical Law of Classical Turbulence

Homogeneous, isotropic, incompressible and steady turbulence

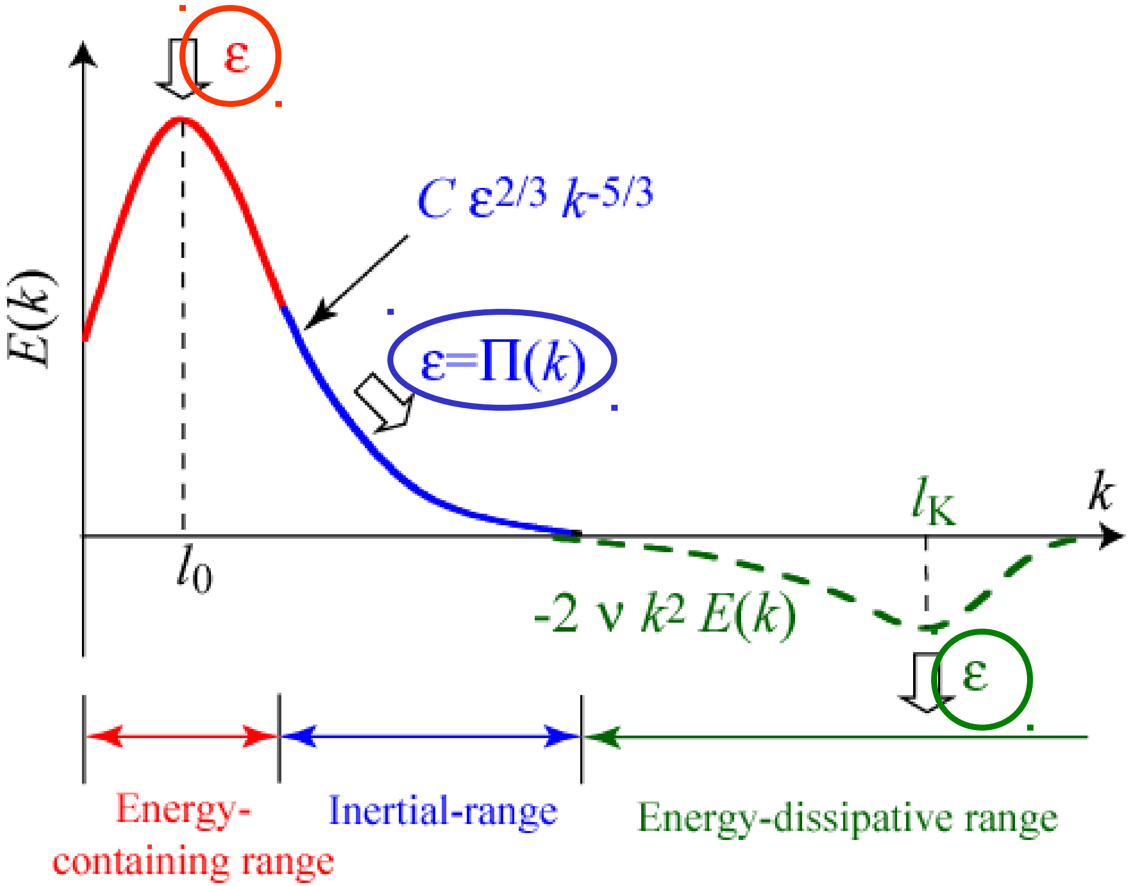


In the energy-dissipative range, energy is dissipated by the viscosity at the Kolmogorov length l_K

$$l_K = \left(\frac{\varepsilon}{\nu^3}\right)^{1/4}$$

Kolmogorov Law : Statistical Law of Classical Turbulence

Homogeneous, isotropic, incompressible and steady turbulence



ε : energy injection rate

ε : energy transportation rate

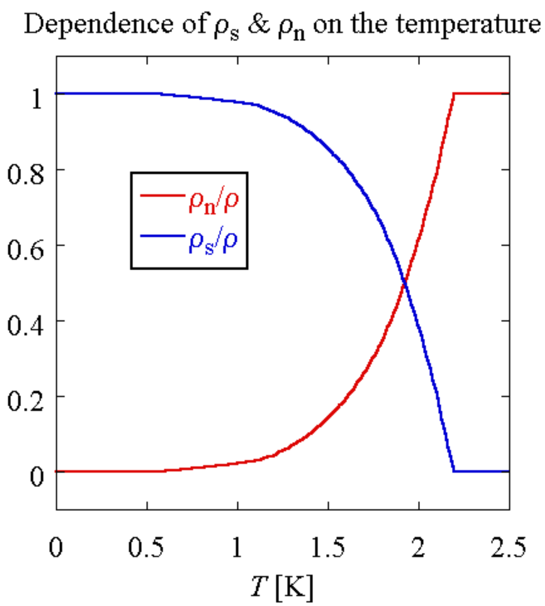
Π(*k*) : energy flux from large to small *k*

ε : energy dissipation rate

What Is The Relation Between Classical and Quantum Turbulence?

Viscous normal fluid + Quantized vortices in inviscid superfluid

Both are coupled together by the friction between normal fluid and quantized vortices (mutual friction) and behave like a conventional fluid

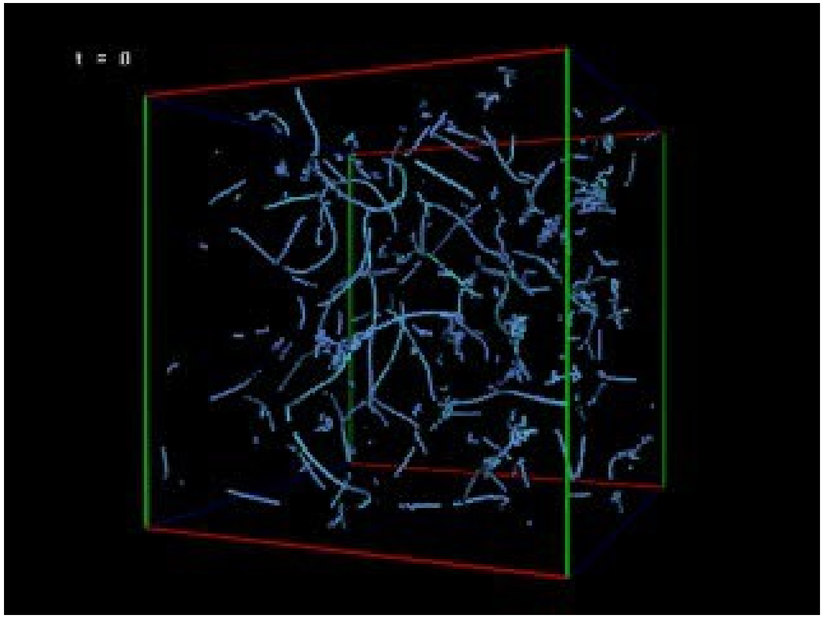


Is there the similarity between classical turbulence and superfluid turbulence without normal fluid (Quantum turbulence)?

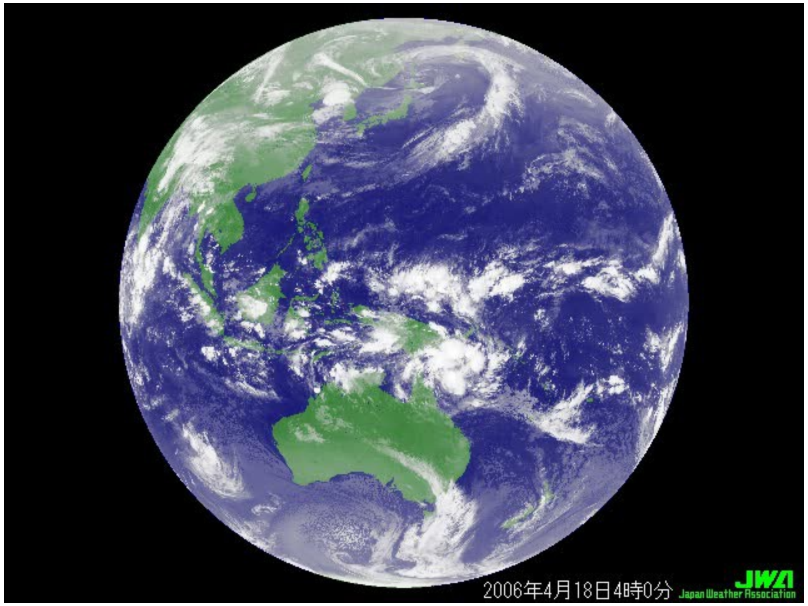
2, Motivation of Studying Quantum Turbulence

Eddies in classical turbulence

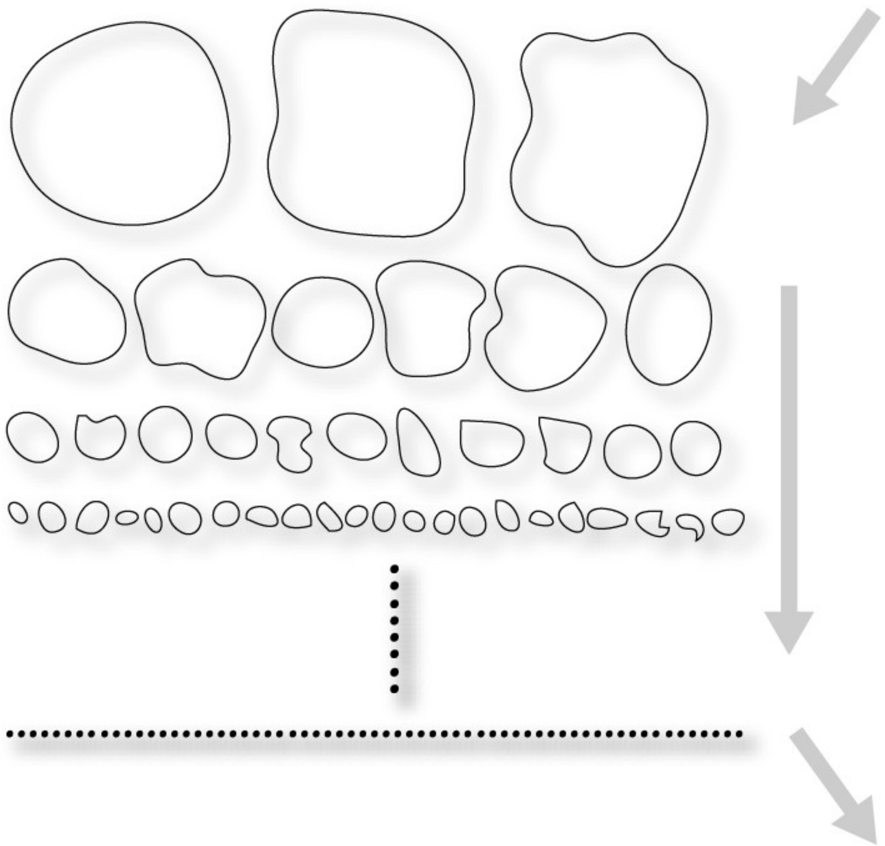
Numerical simulation of NSE (by Kida *et*



Satellite Himawari



Richardson Cascade of Eddies in Classical Turbulence

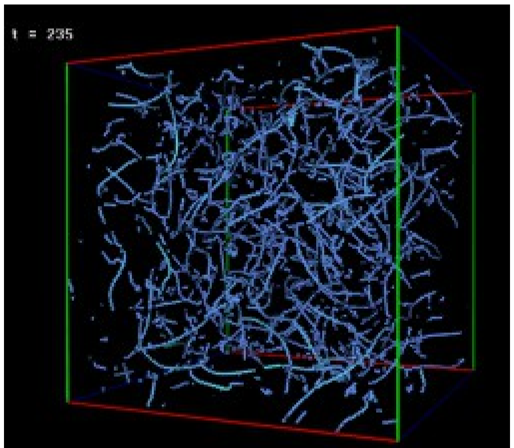


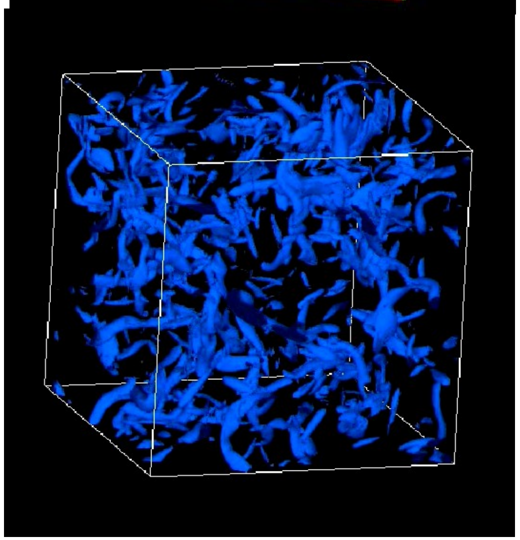
Energy-containing range : generation of large eddies

Inertial-range
Large eddies are broken up to smaller ones in the inertial range : **Richardson cascade**

Energy-dissipative range : disappearance of small eddies

Eddies in Classical Turbulence





•Vorticity $\boldsymbol{\omega} = \mathrm{rot}\ \boldsymbol{v}$ takes continuous value

•Circulation κ becomes arbitrary for arbitrary path.

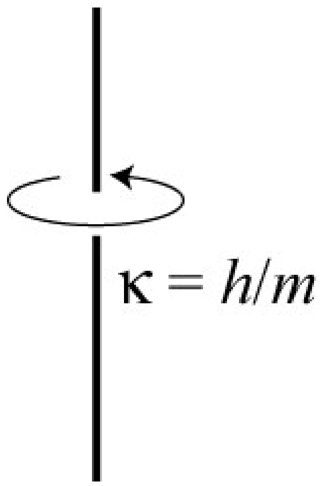
•Eddies are annihilated and nucleated under the viscosity

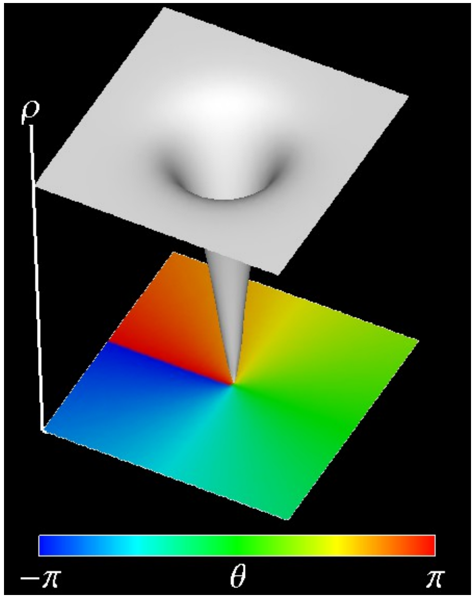
•Definite identification of eddies is difficult.

•The Richardson cascade of eddies is just conceptual (No one had seen the Richardson cascade before).

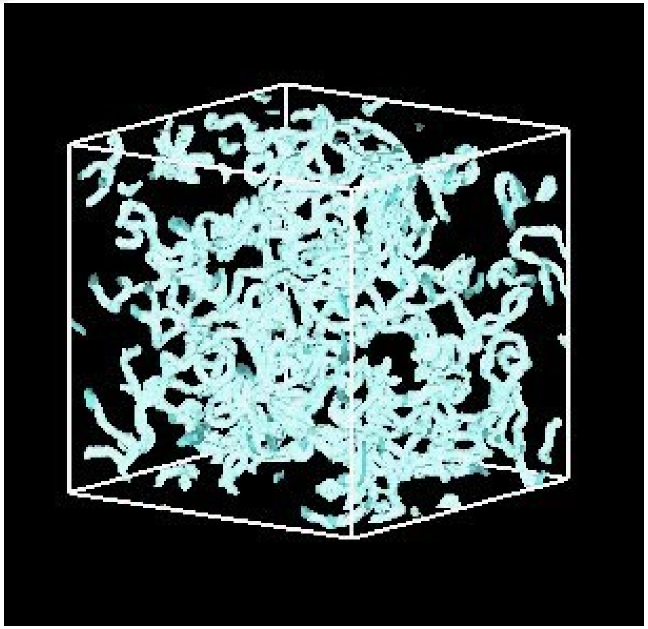
Quantized Vortices in Quantum Turbulence

- Circulation $\kappa = \oint \boldsymbol{v} \cdot \mathrm{d}\boldsymbol{s} = h / m$ around vortex core is quantized.
- Quantized vortex is stable topological defect.
- Vortex core is very thin (the order of the healing length).





Quantum Turbulence



Quantized vortices in superfluid turbulence is definite topological defect

Quantum Turbulence may be able to clarify the relation between the Kolmogorov law and the Richardson cascade!

This Work

1. We study the dynamics and statistics of quantum turbulence by numerically solving the Gross-Pitaevskii equation (with small-scale dissipation).
2. We study the similarity of both decaying and steady (forced) turbulence with classical turbulence.

Model of Gross-Pitaevskii Equation

Numerical simulation of the Gross-Pitaevskii equation

Many boson system

$$\hat{H} = \int \mathrm{d}\boldsymbol{x} \hat{\Psi}^\dagger(\boldsymbol{x},t)[-\nabla^2 + V(\boldsymbol{x}) - \mu + \frac{g}{2}|\hat{\Psi}(\boldsymbol{x},\boldsymbol{t})|^2]|\hat{\Psi}(\boldsymbol{x},t)$$

$$\mathrm{i}\frac{\partial}{\partial t}\hat{\Psi}(\boldsymbol{x},t) = [-\nabla^2 + V(\boldsymbol{x}) - \mu + g\hat{\Psi}^\dagger(\boldsymbol{x},t)\hat{\Psi}(\boldsymbol{x},t)]\hat{\Psi}(\boldsymbol{x},t)$$

$\hat{\Psi}(\boldsymbol{x},t)$: Field operator of bosons

μ : Chemical potential

g : Coupling constant

Model of Gross-Pitaevskii Equation

For Bose-Einstein condensed system

$$\hat{\Psi}(\boldsymbol{x},t) = \Phi(\boldsymbol{x},t) + \hat{\varphi}(\boldsymbol{x},t)$$

$$\Phi(\boldsymbol{x},t) = O(\sqrt{N_0})$$

$$\hat{\varphi}(\boldsymbol{x},t) = O(1) \to 0 \ (\text{at } T=0)$$

$\Phi(\boldsymbol{x},t)$: Macroscopic wave function of BEC

$\hat{\varphi}(\boldsymbol{x},t)$: Quasiparticle fluctuation from BEC

Model of Gross-Pitaevskii Equation

Gross-Pitaevskii equation

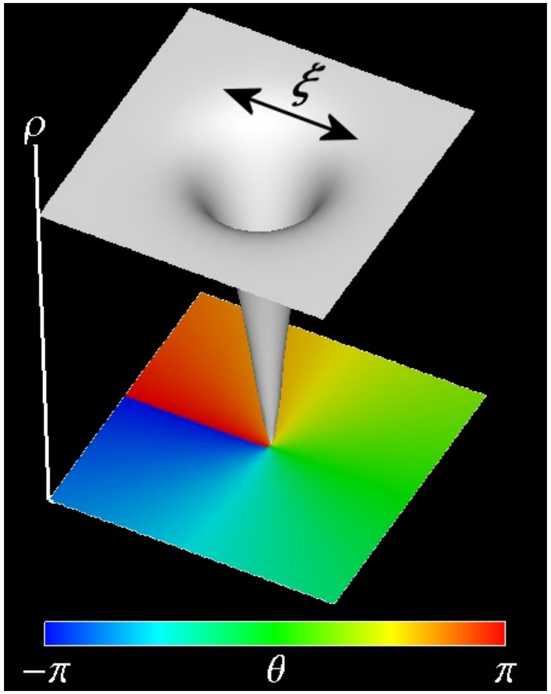
$$\mathrm{i}\frac{\partial}{\partial t}\Phi(\boldsymbol{x},t) = [-\nabla^2 - \mu + g|\Phi(\boldsymbol{x},t)|^2]\Phi(\boldsymbol{x})$$

$$\Phi(\boldsymbol{x}) = |\Phi(\boldsymbol{x})|\exp[\mathrm{i}\theta(\boldsymbol{x})]$$

$$\rho(\boldsymbol{x}) = |\Phi(\boldsymbol{x})|^2 \ : \ \text{Density of fluid}$$

$$\boldsymbol{v}(\boldsymbol{x}) = 2\nabla\theta(\boldsymbol{x}) \ : \ \text{Velocity of fluid}$$

$$\xi = 1/\sqrt{g\bar{\rho}} \ : \ \text{Healing length}$$

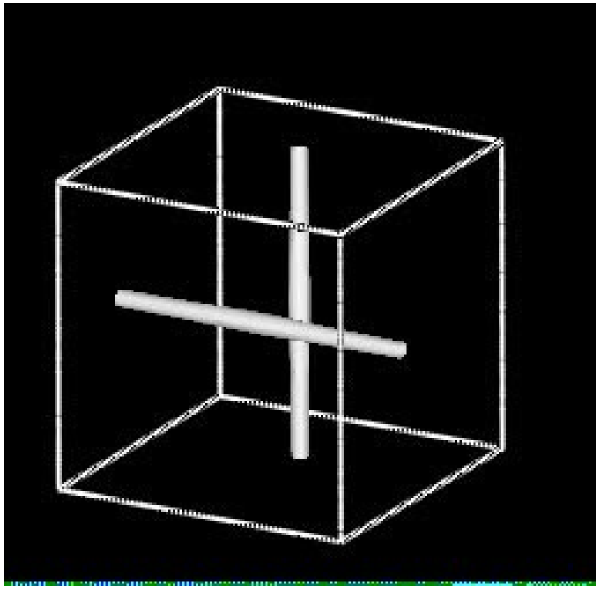


Quantized vortex

We numerically investigate GP turbulence.

Introducing the Dissipation Term

Vortex reconnection



Compressible excitations of wavelength smaller than the healing length are created through vortex reconnections and through the disappearance of small vortex loops.

→Those excitations hinder the cascade process of quantized vortices!

Introducing the Dissipation Term

To remove the compressible short-wavelength excitations, we introduce a small-scale dissipation term into GP equation

Fourier transformed GP equation

$$\mathrm{i}\frac{\partial \Phi(k)}{\partial t} = \left[(k^2-\mu)\Phi(k) + \frac{g}{V^2}\sum_{k_1,k_2}\Phi(k_1)\Phi^*(k_2)\Phi(k-k_1+k_2)\right]$$

$$[\mathrm{i}-\gamma(k)]\frac{\partial \Phi(k)}{\partial t} = \left[(k^2-\mu)\Phi(k) + \frac{g}{V^2}\sum_{k_1,k_2}\Phi(k_1)\Phi^*(k_2)\Phi(k-k_1+k_2)\right]$$

$$\gamma(k) = \gamma_0\theta(k-2\pi/\xi)\ :\ \text{smaller scale dissipation than } \xi$$

4, Numerical Results -Decaying Turbulence-

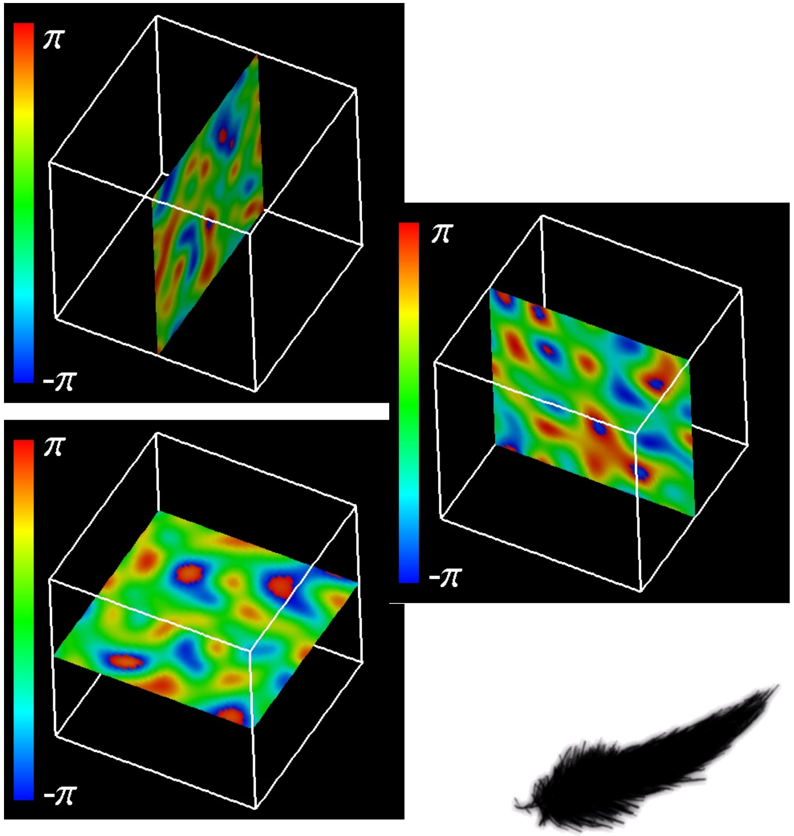
Initial state : random phase

$$\Phi = \sqrt{\rho}\exp(\mathrm{i}\theta)$$
$$\rho(t=0)\ :\ \mathrm{uniform}$$
$$\theta(t=0)\ \mathrm{random\ in\ space}$$

Initial velocity : random

↓

Turbulence is created



Decaying Turbulence

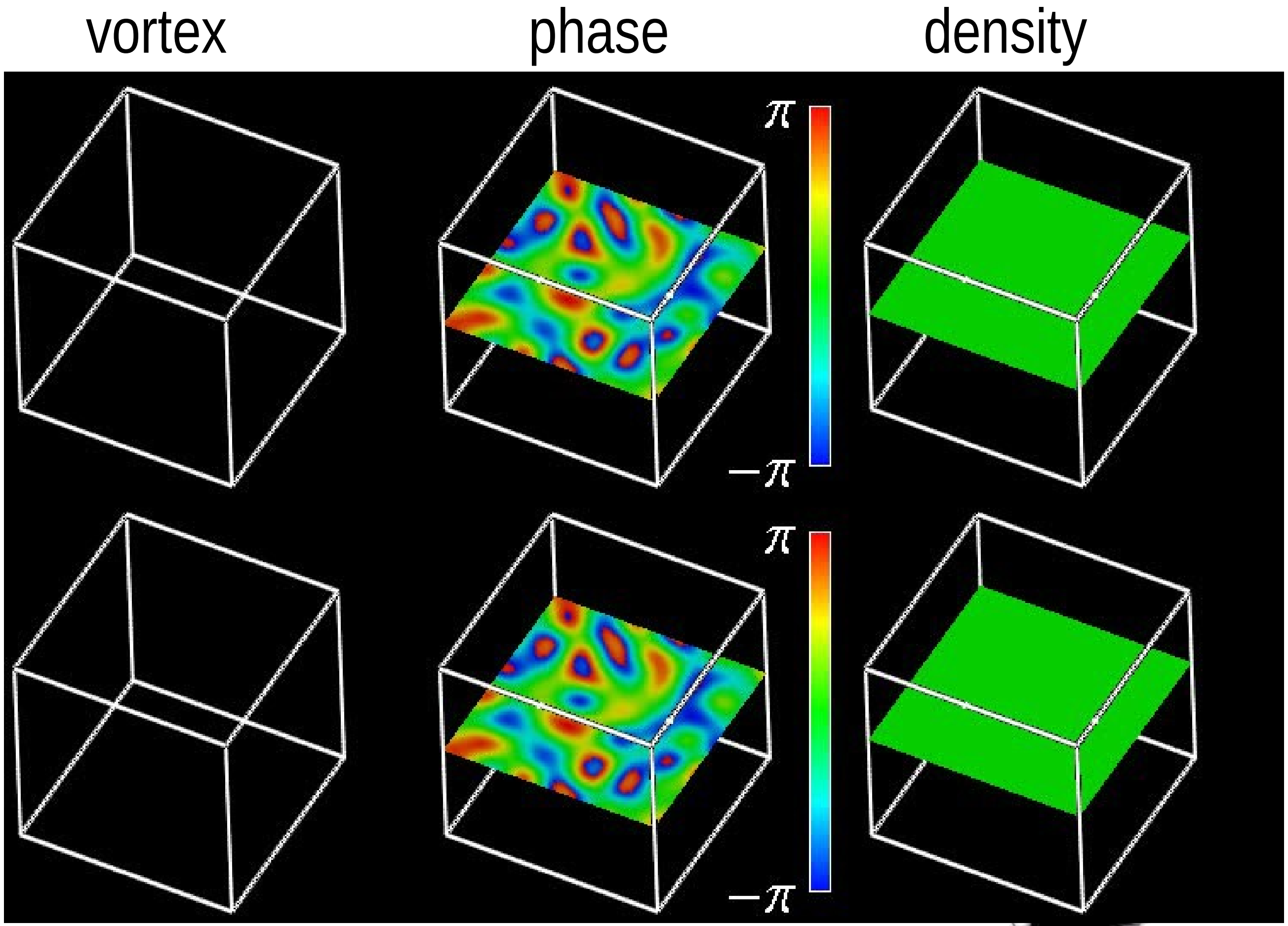
$0 < t < 6$

$\gamma_0 = 0$

without dissipation

$\gamma_0 = 1$

with dissipation



Decaying Turbulence

Calculating kinetic energy of vortices and compressible excitations

$$E_{\text{kin}} = \int \mathrm{d}\boldsymbol{x}\, [|\Phi(\boldsymbol{x})|\nabla\theta(\boldsymbol{x})]^2$$

Kinetic energy

$$E^{\text{i}}_{\text{kin}} = \int \mathrm{d}\boldsymbol{x}\, [\{|\Phi(\boldsymbol{x})|\nabla\theta(\boldsymbol{x})\}^{\text{i}}]^2 \quad \text{div}\{|\Phi(\boldsymbol{x})|\nabla\theta(\boldsymbol{x})\}^{\text{i}} = 0$$

Incompressible kinetic energy (vortex)

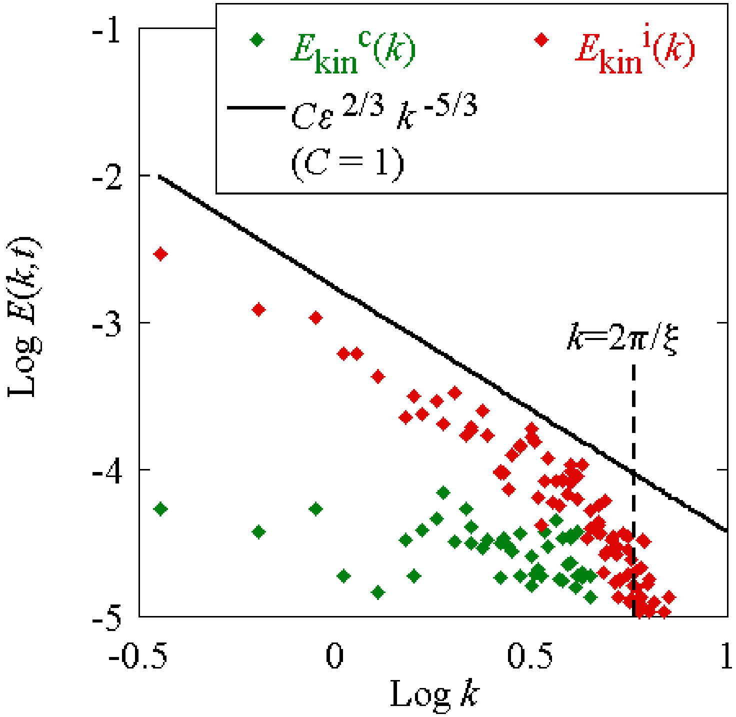
$$E^{\text{c}}_{\text{kin}} = \int \mathrm{d}\boldsymbol{x}\, [\{|\Phi(\boldsymbol{x})|\nabla\theta(\boldsymbol{x})\}^{\text{c}}]^2 \quad \text{rot}\{|\Phi(\boldsymbol{x})|\nabla\theta(\boldsymbol{x})\}^{\text{c}} = 0$$

Compressible kinetic energy (compressible excitation)

$$E_{\text{kin}} = E^{\text{i}}_{\text{kin}} + E^{\text{c}}_{\text{kin}}$$

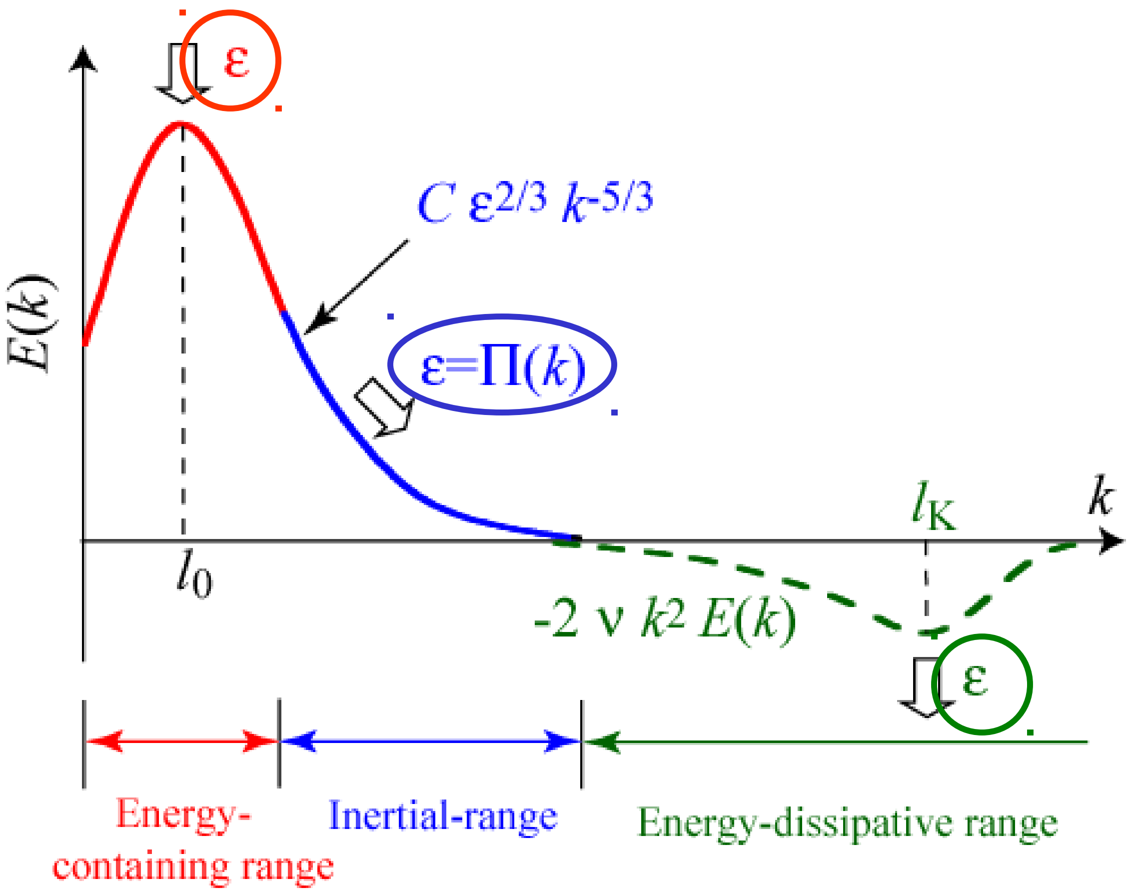
Energy Spectrum of Decaying Turbulence

$$\text{Energy spectrum}: \; E_{\rm kin}^{\rm i,c} = \int \mathrm{d}k \; E_{\rm kin}^{\rm i,c}(k)$$



Quantized vortices in quantum turbulence show the similarity with classical turbulence

Numerical Results -Steady Turbulence-



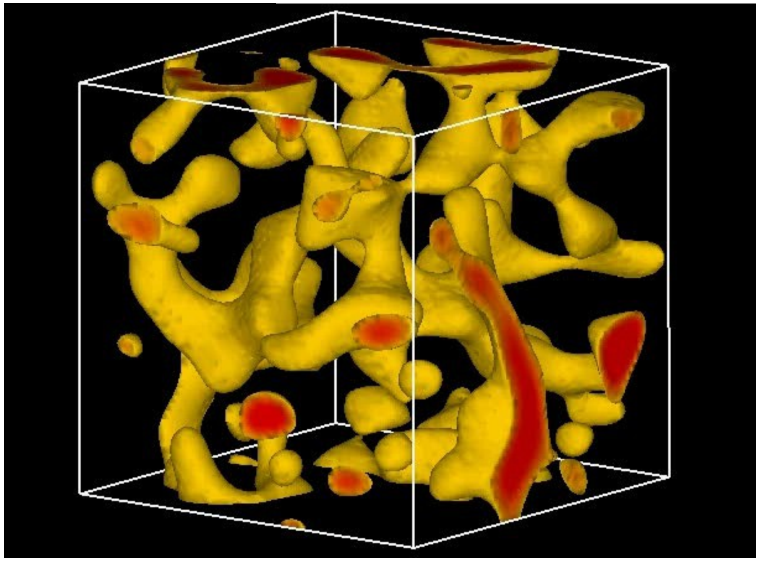
Steady turbulence with the energy injection enables us to study detailed statistics of quantum turbulence.

Energy Injection As Moving Random Potential

$$[\mathrm{i}-\gamma(\boldsymbol{x},t)]\frac{\partial}{\partial t}\Phi(\boldsymbol{x},t)=[-\nabla^2-\mu+U(\boldsymbol{x},t)+g|\Phi(\boldsymbol{x},t)|^2]\Phi(\boldsymbol{x})$$

$$U(\boldsymbol{x}) : \text{Moving random potential}$$

$$\langle U(\boldsymbol{x},t)U(\boldsymbol{x}',t')\rangle = V_0^2 \exp\left[-\frac{(\boldsymbol{x}-\boldsymbol{x}')^2}{2X_0^2}-\frac{(t-t')^2}{2T_0^2}\right]$$



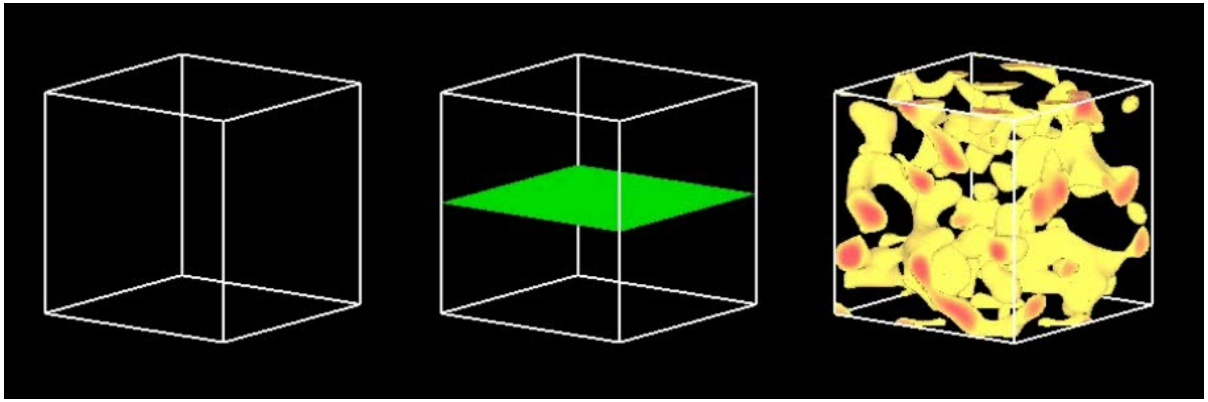
X_0 : characteristic scale of the moving random potential

$\rightarrow$Vortices of radius X_0 are nucleated

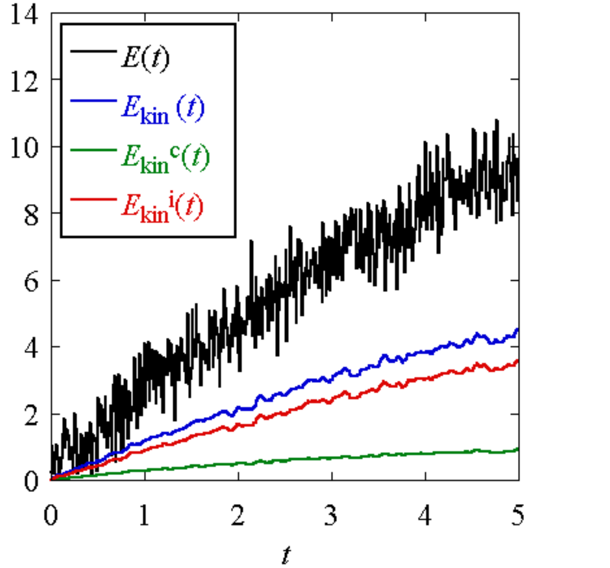
Steady Turbulence

Steady turbulence is realized by the competition between energy injection and energy dissipation

vortex density potential



Time dependence of kinetic energy at initial stage

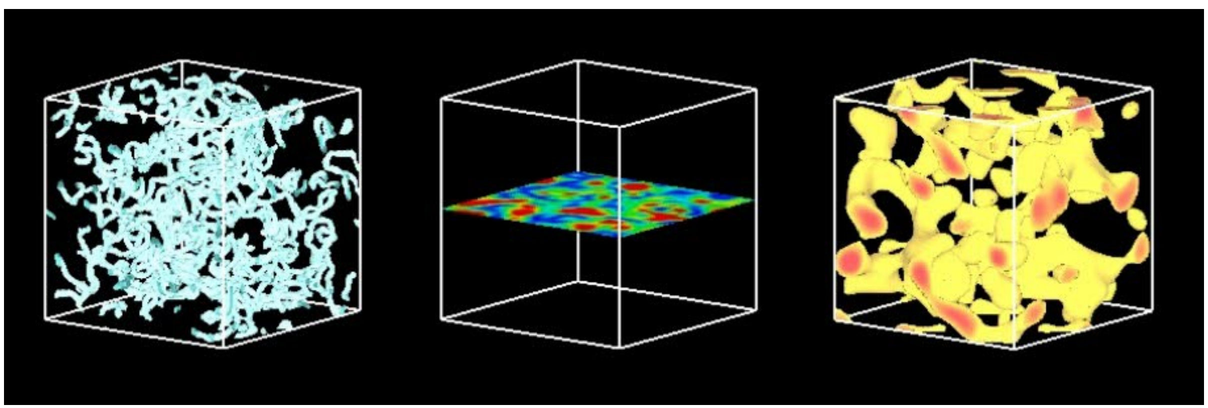


Energy of vortices $E_{\mathrm{kin}}^{\mathrm{i}}$ is always dominant

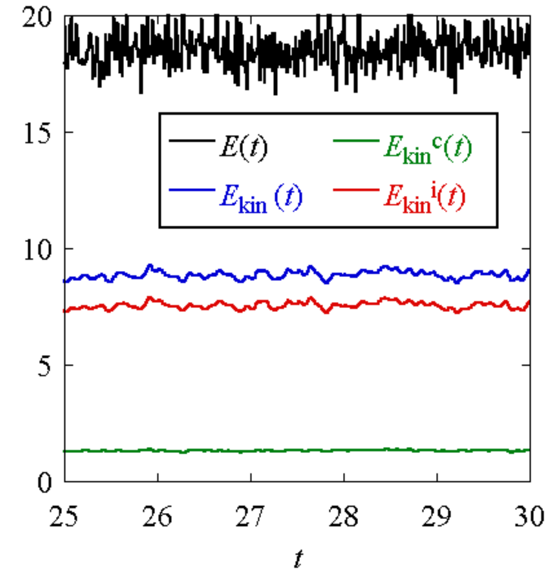
Steady Turbulence

Steady turbulence is realized by the competition between energy injection and energy dissipation

vortex density potential

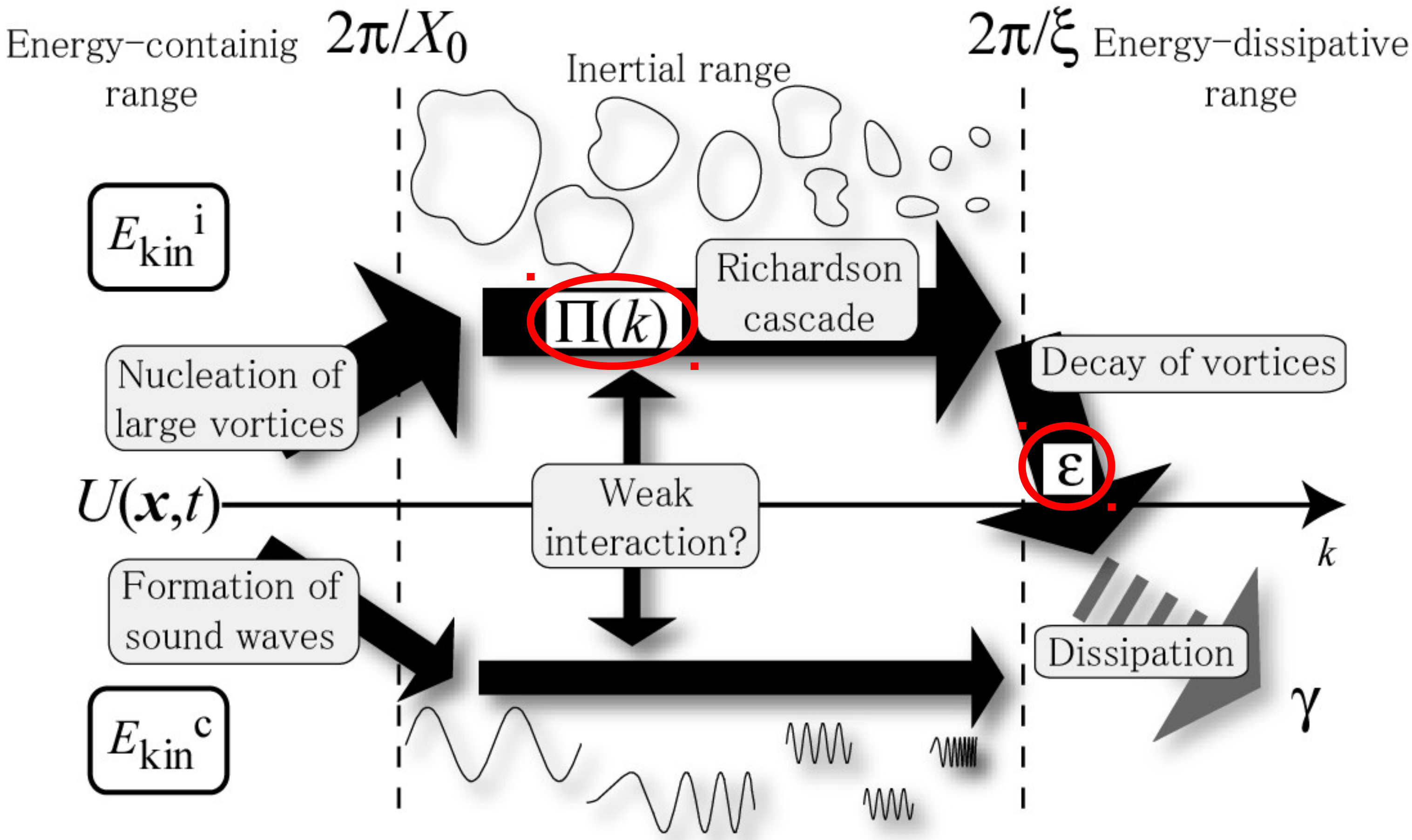


Time dependence of kinetic energy at steady stage



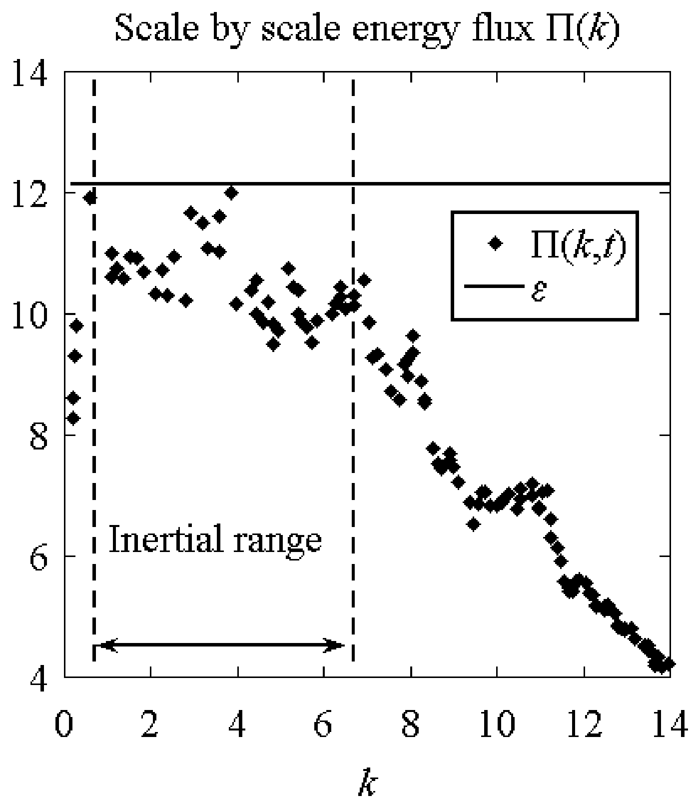
Energy of vortices E_{kin}^{i} is always dominant

Flow of Energy in Steady Quantum Turbulence



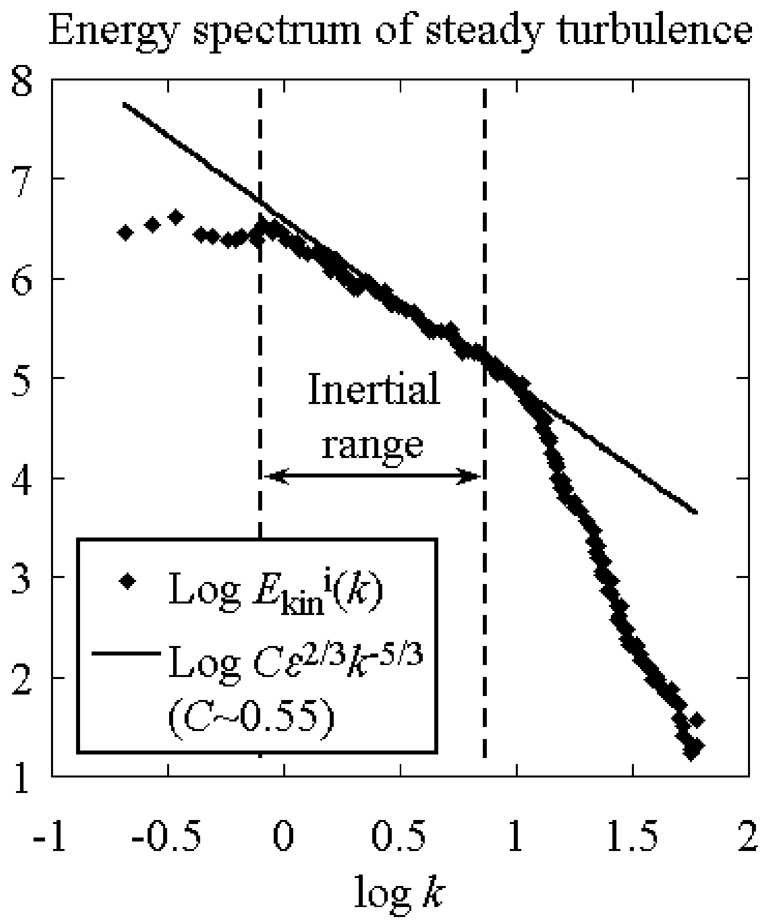
Energy Dissipation Rate and Energy Flux

Energy flux $\Pi(k)$ is obtained by the energy budget equation from the GP equation.



1. $\Pi(k)$ is almost constant in the inertial range
2. $\Pi(k)$ in the inertial range is consistent with the energy dissipation rate ε

Energy Spectrum of Steady Turbulence



Energy spectrum shows the Kolmogorov law again

→ Similarity between quantum and classical turbulence is supported!

5, Summary

1. We did the numerical simulation of quantum turbulence by numerically solving the Gross-Pitaevskii equation.
2. We succeeded clarifying the similarity between classical and quantum turbulence.
3. We also clarify the flow of energy in quantum turbulence by calculating the energy dissipation rate and the energy flux in steady turbulence.

Future Outlook of Quantum Turbulence

Quantum mechanics and quantum turbulence

Classical turbulence and quantum turbulence are in different fields of physics from now.

↓

It is probed that quantum turbulence can become a ideal prototype to understand turbulence in the aspect of vortices.

→ New breakthrough for understanding turbulence

Quantum Turbulence : Past Simulation

T. Araki, M. Tsubota and S. K. Nemirovskii, Phys. Rev. Lett. **89**, 145301 (2002)

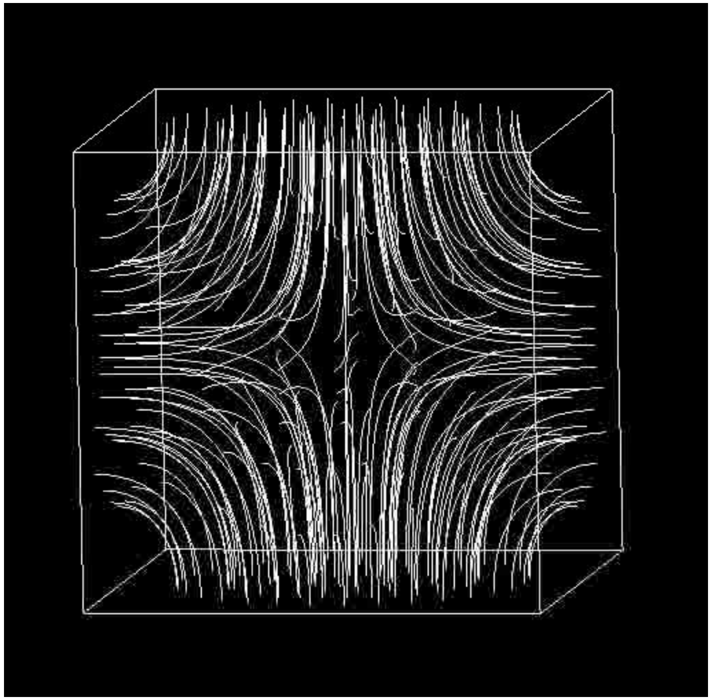
Calculate the energy spectrum of quantum turbulence by using the vortex filament model (initial condition : Taylor-Green-flow)

$$\frac{\partial \boldsymbol{x}_0(t)}{\partial t} = \boldsymbol{v}_{\mathrm{s}}(\boldsymbol{x}_0)$$

$$\boldsymbol{v}_{\mathrm{s}}(\boldsymbol{x}) = \boldsymbol{v}_{\mathrm{ind}}(\boldsymbol{x}) + \boldsymbol{v}_{\mathrm{sa}}(\boldsymbol{x})$$

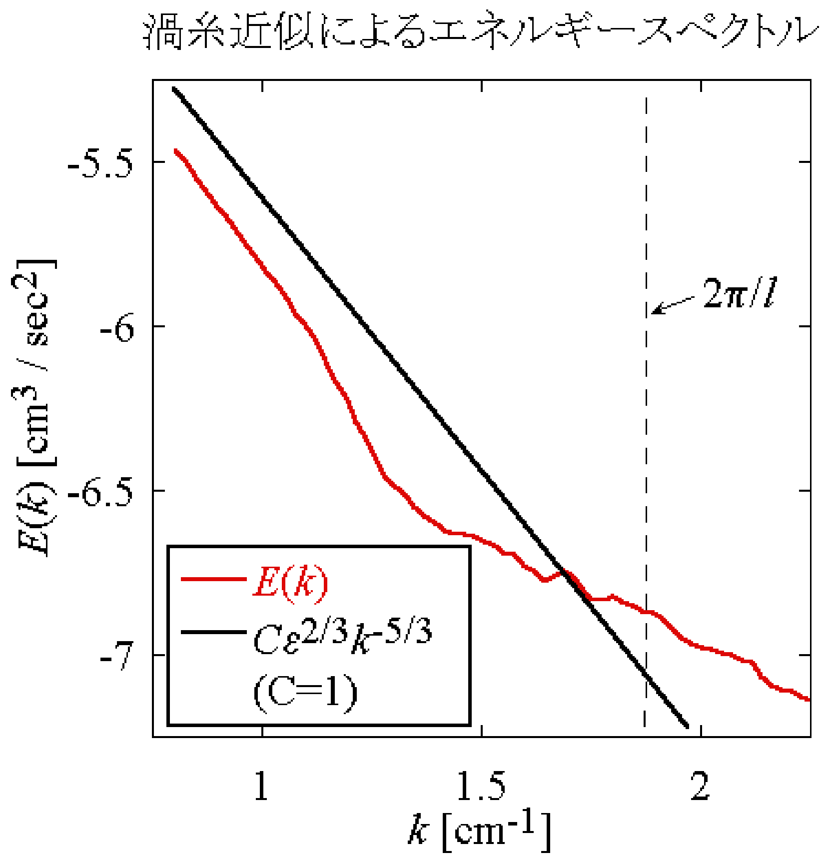
$$\boldsymbol{v}_{\mathrm{ind}}(\boldsymbol{x}) = \frac{\kappa}{4\pi}\int \frac{[\boldsymbol{x}_0(t) - \boldsymbol{x}] \times \mathrm{d}\boldsymbol{x}_0(t)}{|\boldsymbol{x}_0(t) - \boldsymbol{x}|^3}$$

No mutual friction



Solid boundary condition

Quantum Turbulence : Past Simulation



Energy spectrum is consistent with the Kolmogorov law at low k ($C \fallingdotseq 0.7$)

Simulation of Quantum Turbulence : Numerical Parameters

Length scale is normalized by the lealing length ξ.

System is periodic box with 256^3 grid points

Spatial resolution $\Delta x = 1/8$: ξ includes 8 grid points

Volume of system $V = 32^3$

Wavenumber resolution $\Delta k = 2\pi/32 = 0.196$

Coupling constant $g = 1$

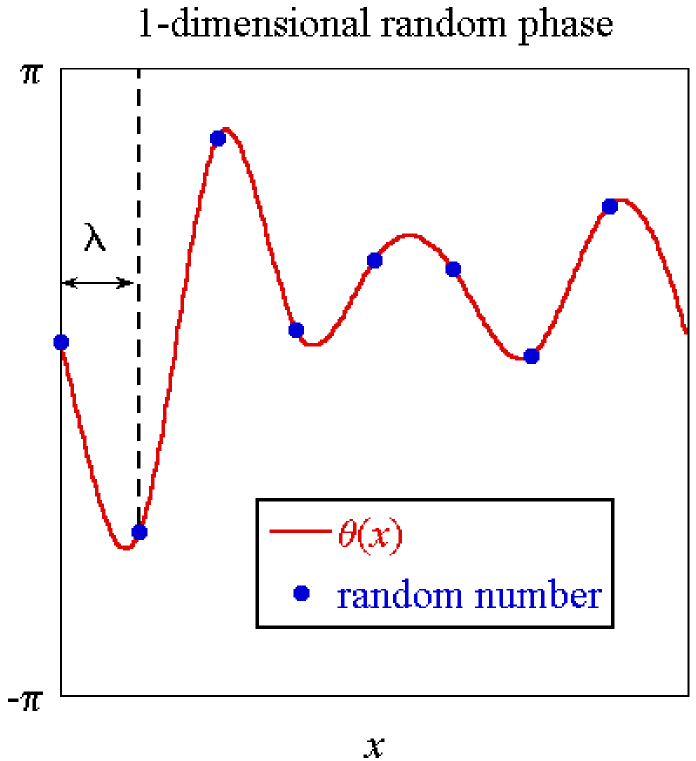
Time resolution $\Delta t = 0.0001$

Space : Pseudo-spectral method

Time : Runge-Kutta-Verner method

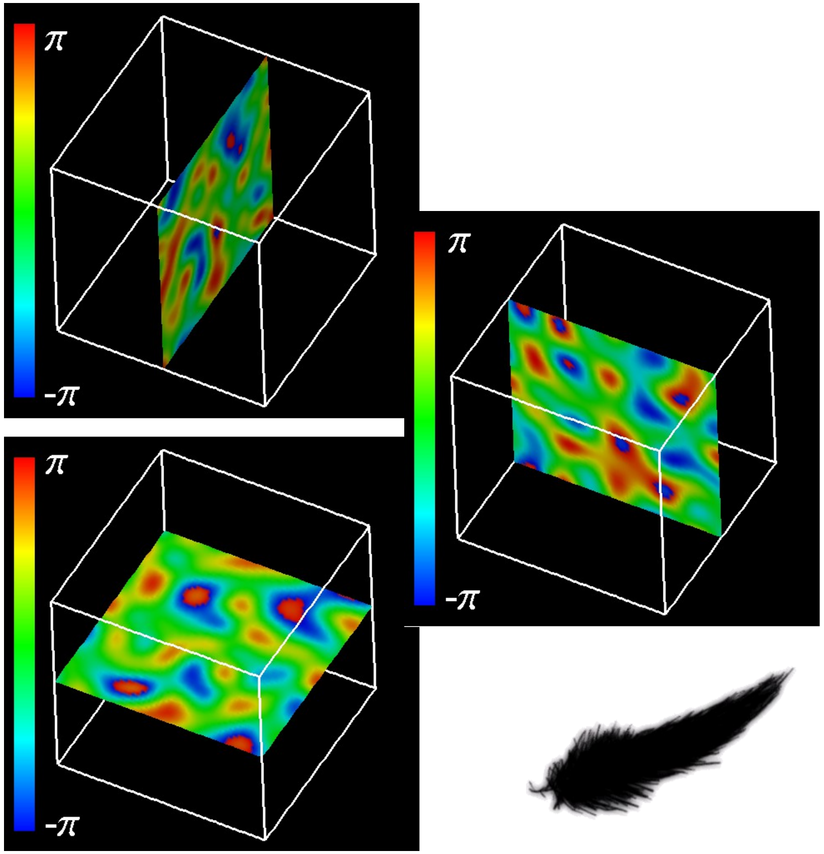
Simulation of Quantum Turbulence : 1, Decaying Turbulence

There is no energy injection and the initial state has random phase.



3D

$\lambda = 4$



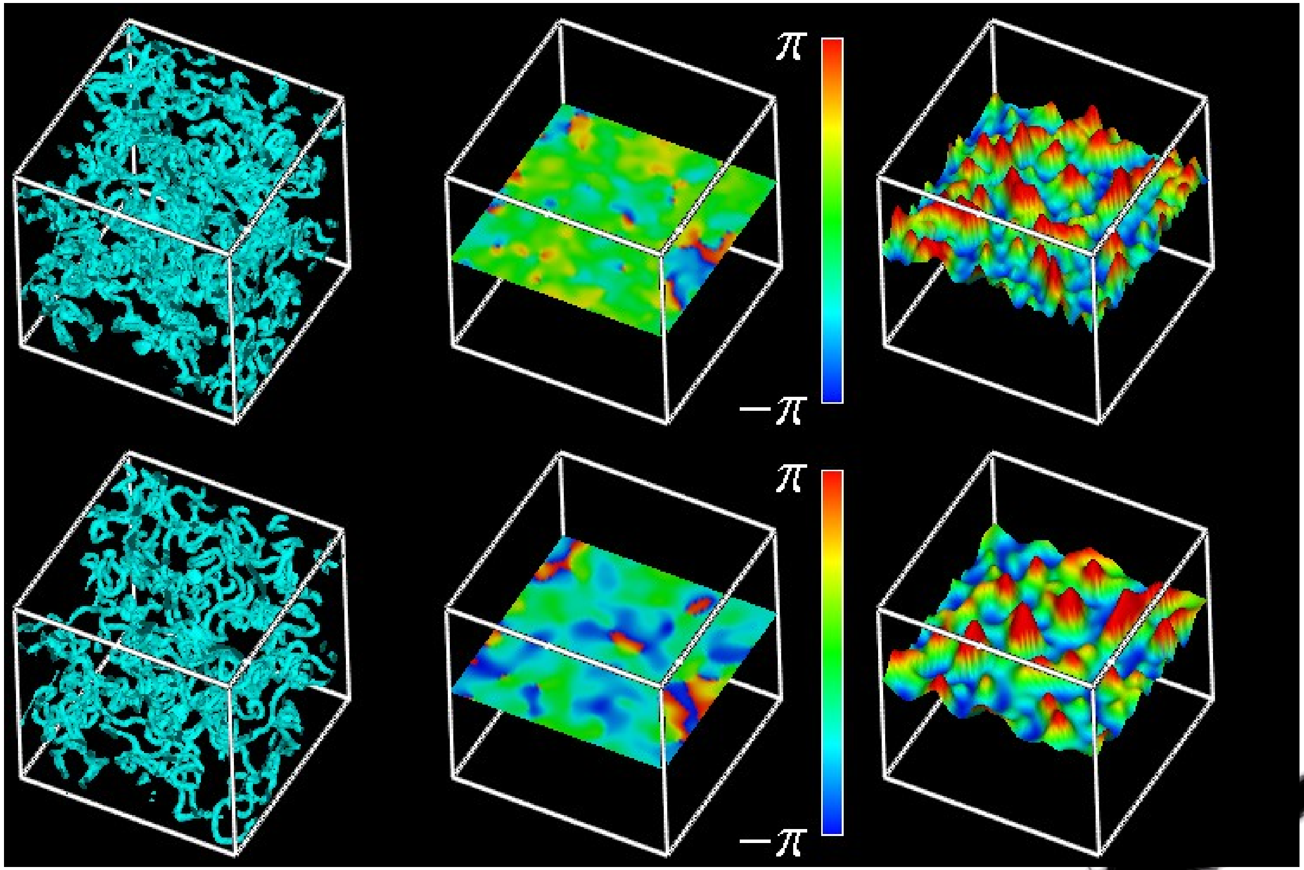
Decaying Turbulence

$t = 5$

vortex | phase | density

$\gamma_0 = 0$

$\gamma_0 = 1$

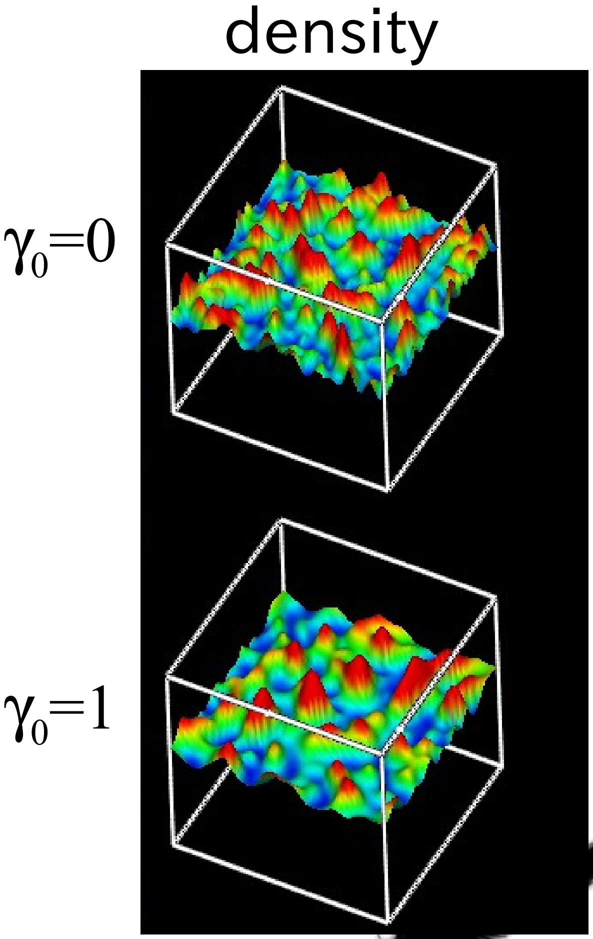


Decaying Turbulence

$t = 5$

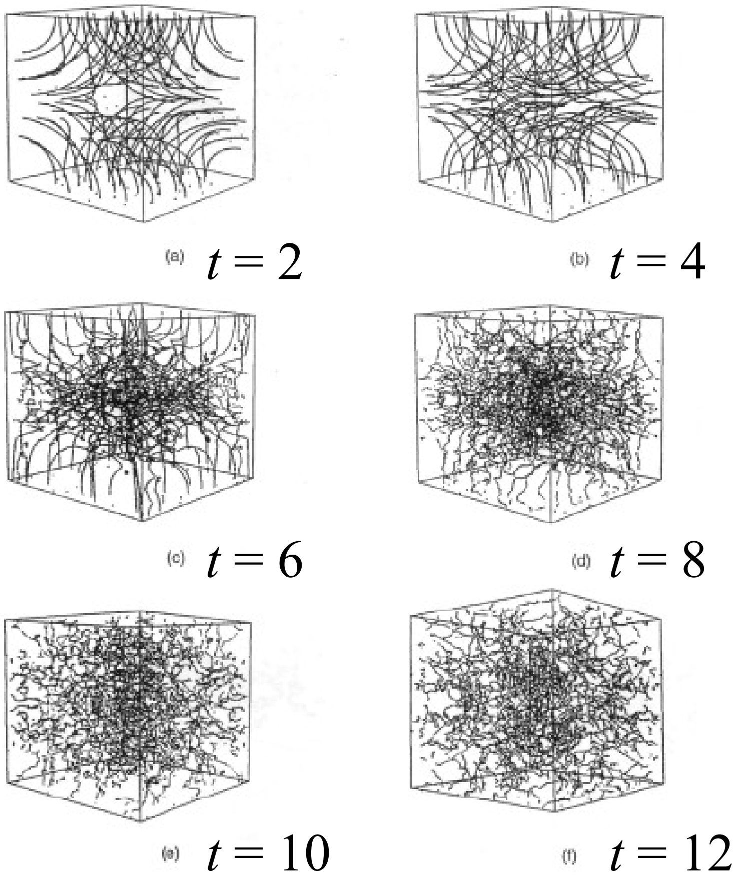
Small structures in $\gamma_0 = 0$ are dissipated in $\gamma_0 = 1$

→Dissipation term dissipates only short-wavelength excitations.



Without Dissipating Compressible Excitations…

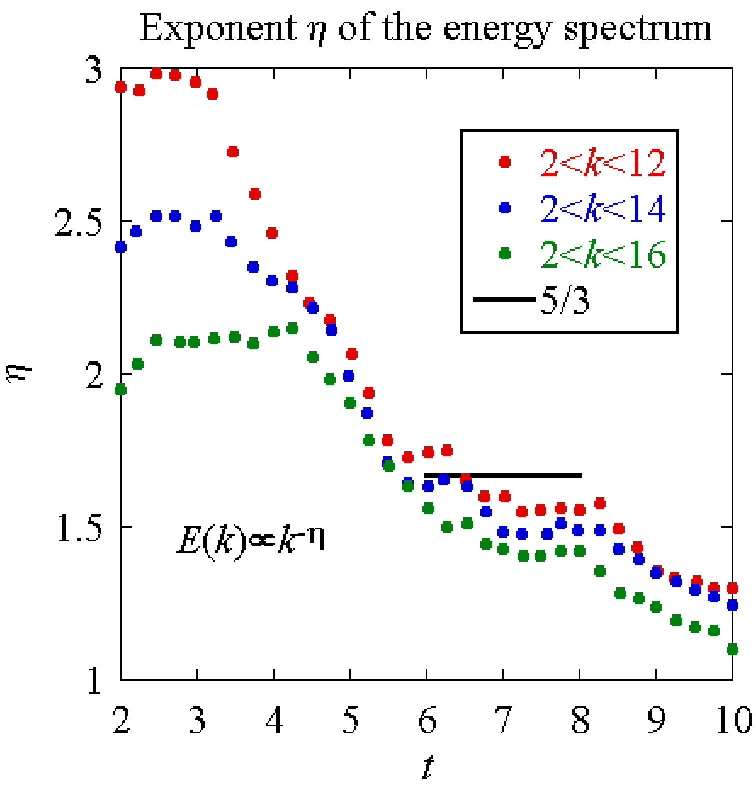
C. Nore, M. Abid, and M. E. Brachet, Phys. Rev. Lett. **78**, 3896



Numerical simulation of GP turbulence

The incompressible kinetic energy changes to compressible kinetic energy while conserving the total energy

Without Dissipating Compressible Excitations…

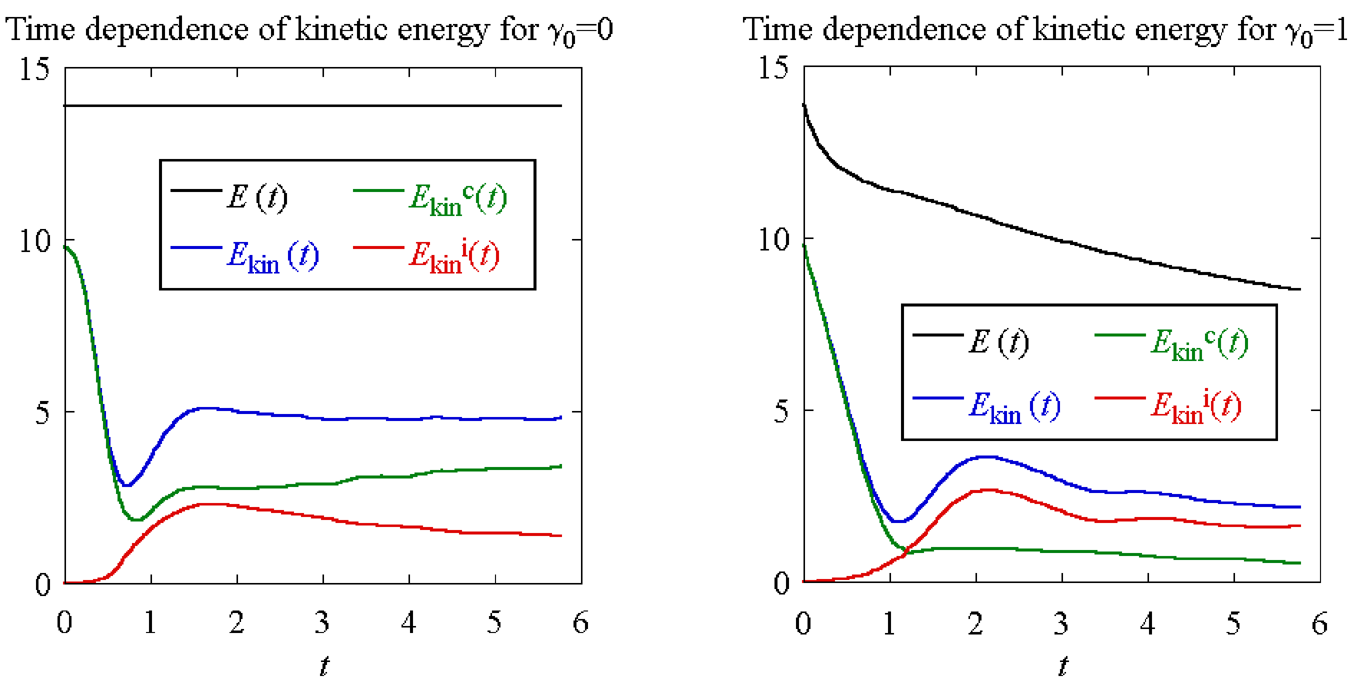


The energy spectrum is consistent with the Kolmogorov law in a short period

→This consistency is broken in late stage with many compressible excitations

We need to dissipate compressible excitations

Decaying Turbulence

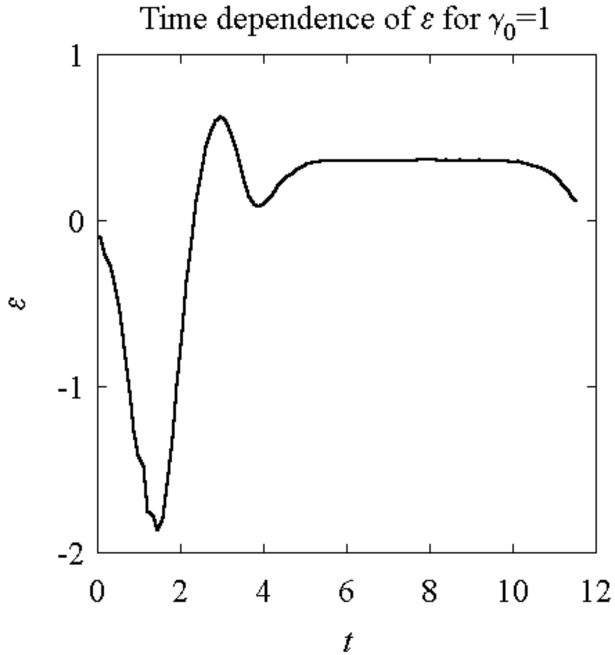


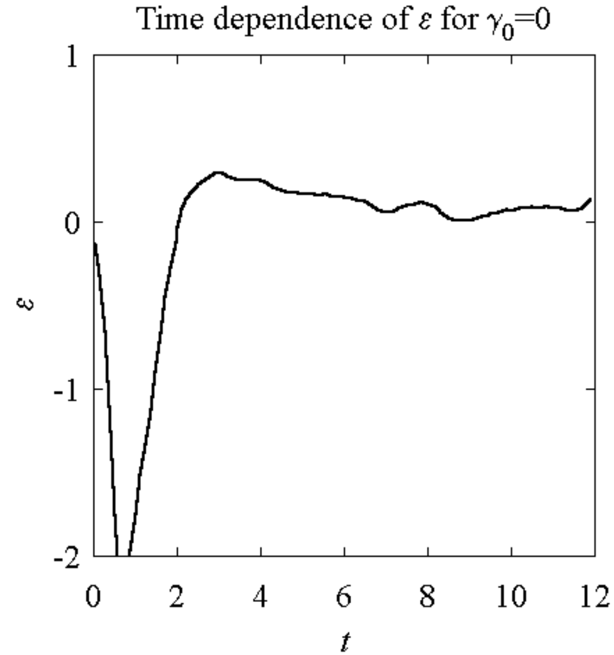
$\gamma_0 = 0$: Energy of compressible excitations $E_{kin}^{\,c}$ is dominant

$\gamma_0 = 1$: Energy of vortices $E_{kin}^{\,i}$ is dominant

Comparison With Classical Turbulence : Energy Dissipation Rate

Energy dissipation rate of vortices : $\varepsilon = -\partial E^{\mathrm{i}}_{\mathrm{kin}}/\partial t$



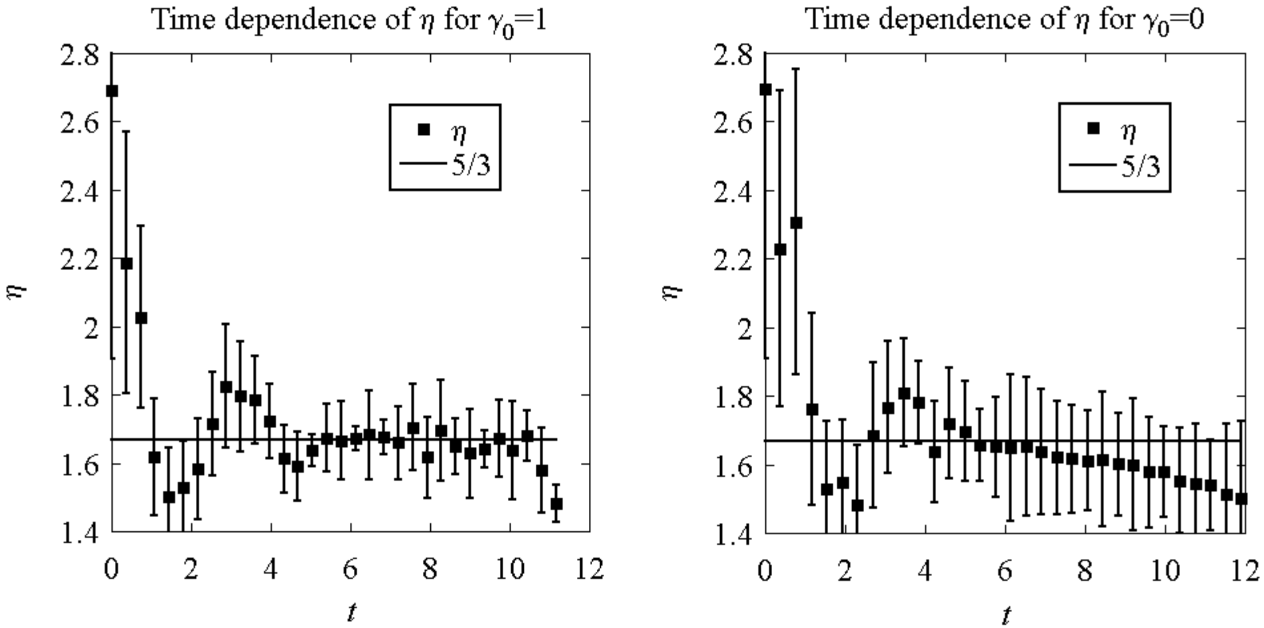


$\gamma_0 = 1$: ε is almost constant at $4 < t < 10$ (quasi steady state)

$\gamma_0 = 0$: ε is unsteady (Interaction with compressible excitations)

Comparison With Classical Turbulence : Energy Spectrum

$$\text{Exponent } \eta \text{ of energy spectrum : } E^{\mathrm{i}}_{\mathrm{kin}} = \int \mathrm{d}k \, E^{\mathrm{i}}_{\mathrm{kin}}(k) \quad E^{\mathrm{i}}_{\mathrm{kin}}(k) \propto k^{-\eta}$$



Straight line fitting at $\Delta k < k < 2\pi/\xi$

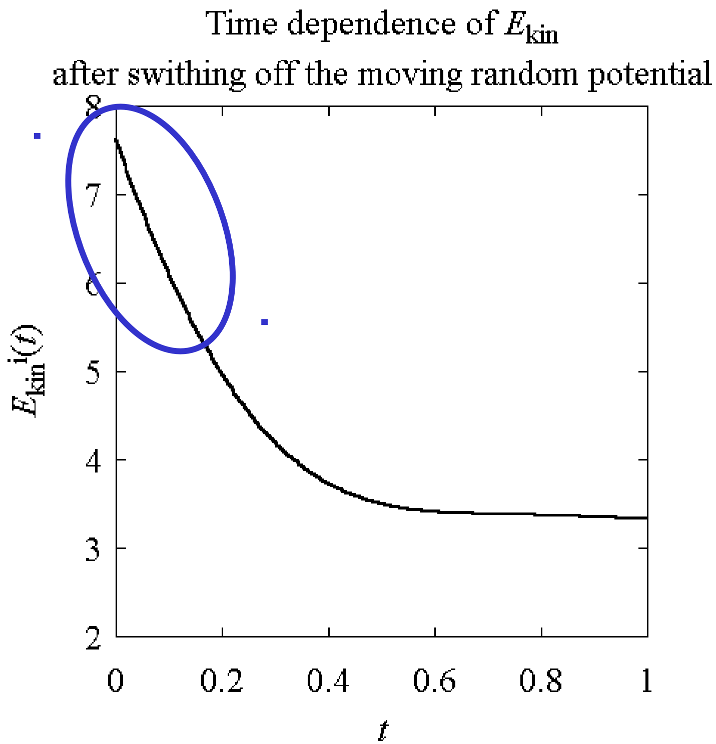
: Non-dissipating range

$\gamma_0 = 1$: η = -5/3 at $4 < t < 10$

$\gamma_0 = 0$: η = -5/3 at $4 < t < 7$

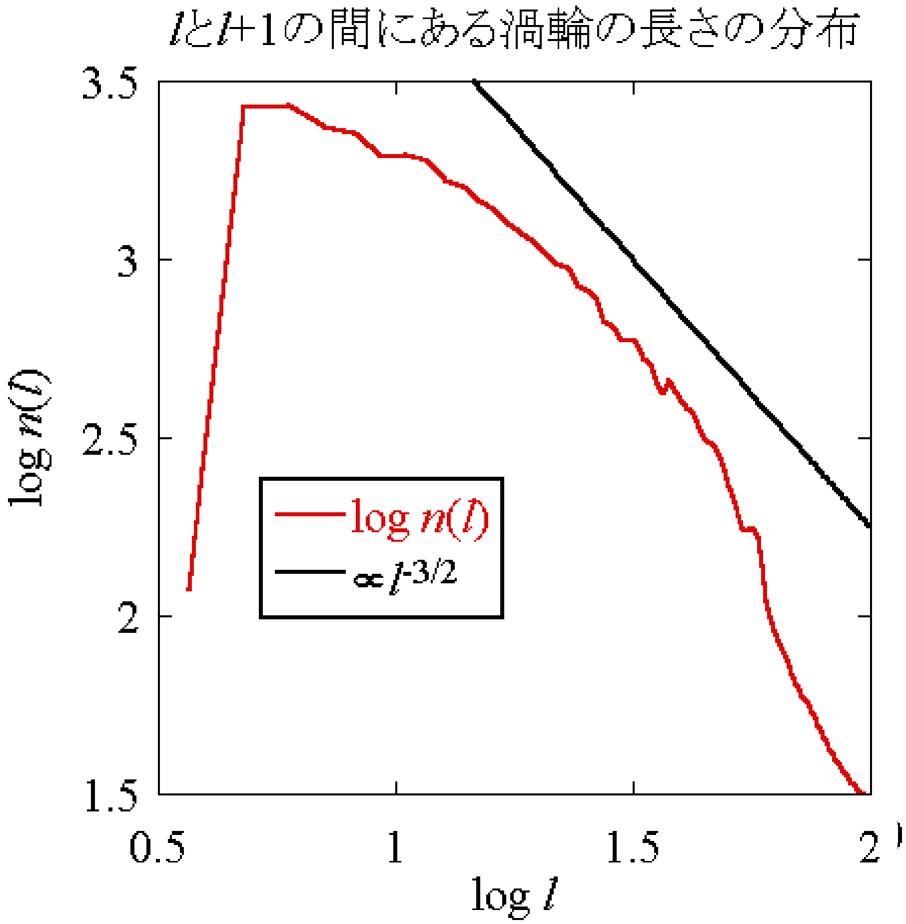
Energy Dissipation Rate and Energy Flux

Energy dissipation rate ε is obtained by switching off the moving random potential



$$\varepsilon = -\frac{\partial E_{\mathrm{kin}}^{\mathrm{i}}}{\partial t} = 12.5$$

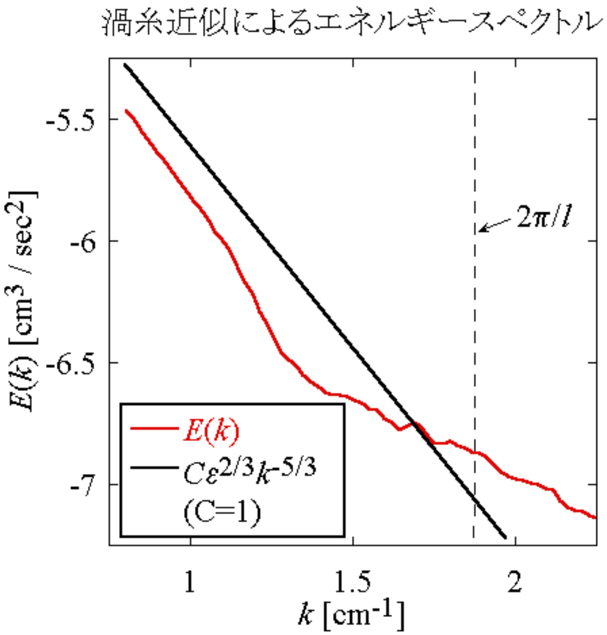
Vortex Size Distribution

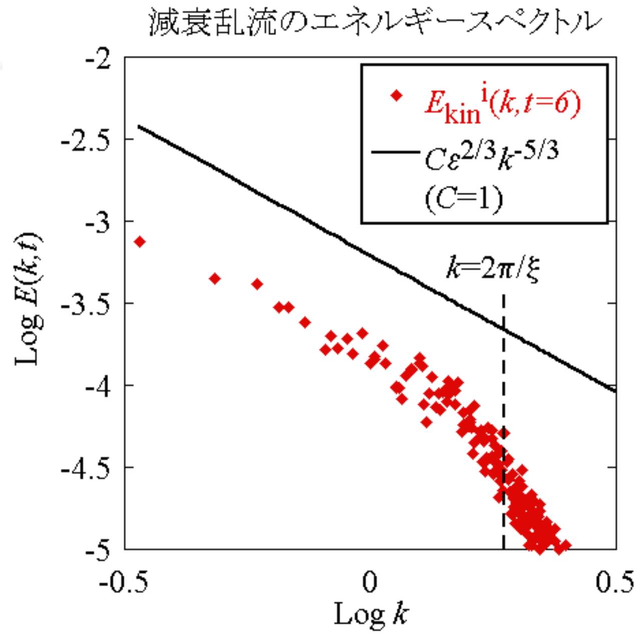


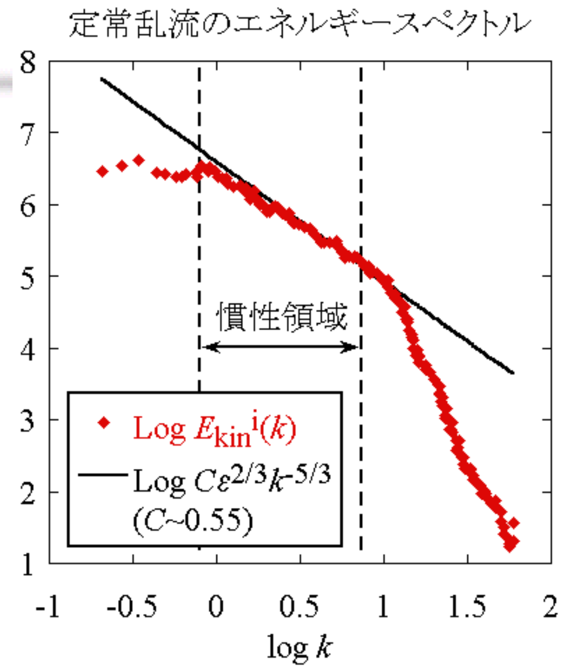
$n(l) \propto l^{-3/2}$

?

Kolmogorov Constant







Vortex filament : $C \sim 0.7$

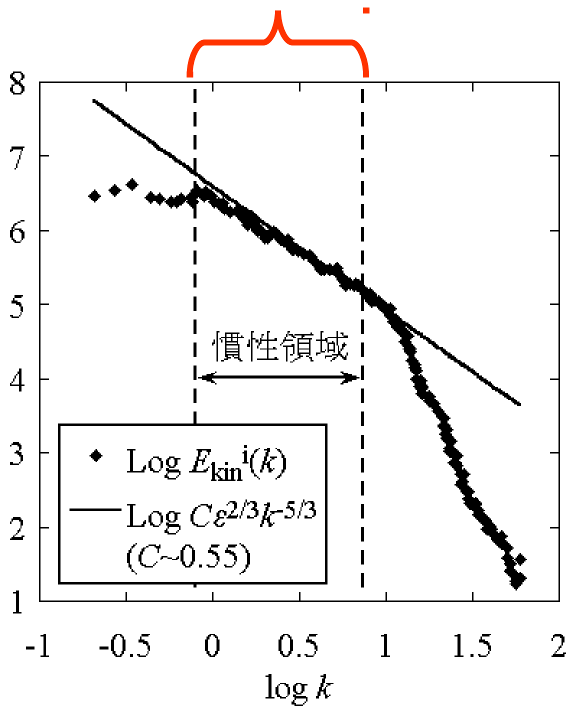
Decaying turbulence : $C \sim 0.32$

Steady turbulence : $C \sim 0.55$

Classical turbulence : $1.4 < C < 1.8$ → Smaller than classical Kolmogorov constant (It may be characteristic in quantum turbulence)

Extension of the Inertial Range

Depend on the scale of simulation

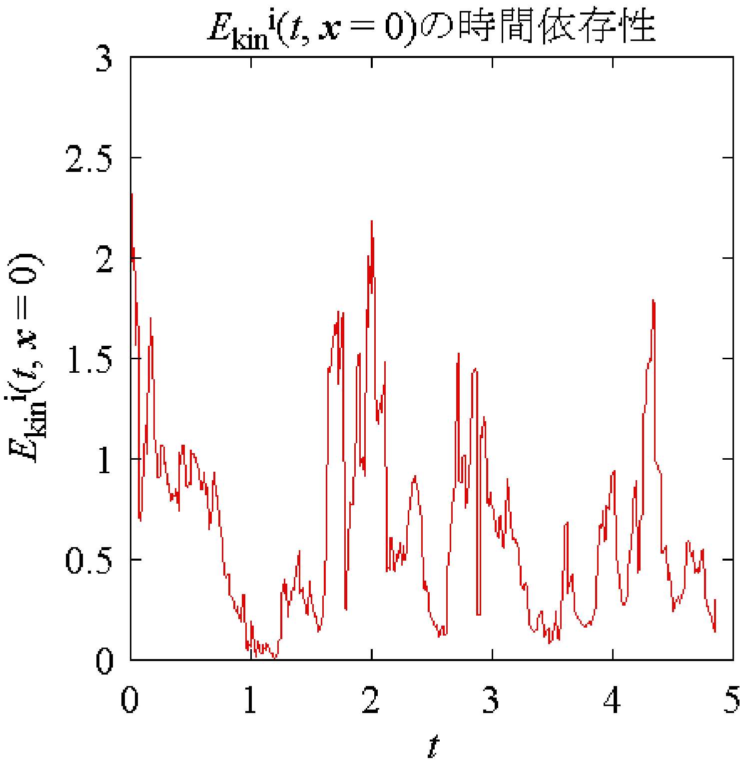


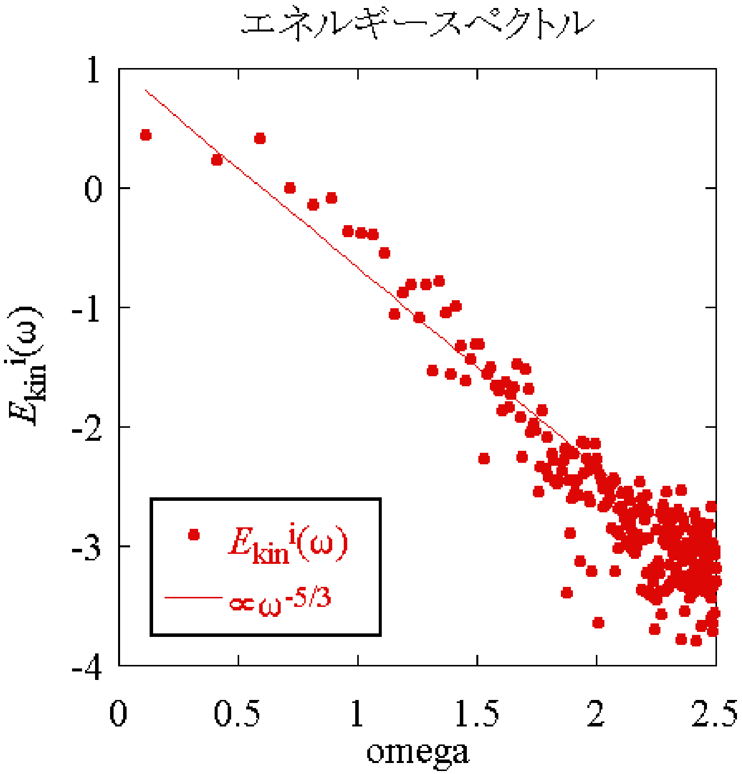
Energy spectrum for time correlation

$$E^{\mathrm{i}}_{\mathrm{kin}}(t) = \int \mathrm{d}\boldsymbol{x}\; E^{\mathrm{i}}_{\mathrm{kin}}(\boldsymbol{x}, t) = \int \mathrm{d}k\; E^{\mathrm{i}}_{\mathrm{kin}}(k, t)$$

$$E^{\mathrm{i}}_{\mathrm{kin}}(\boldsymbol{x}) = \int \mathrm{d}t\; E^{\mathrm{i}}_{\mathrm{kin}}(\boldsymbol{x}, t) = \int \mathrm{d}\omega\; E^{\mathrm{i}}_{\mathrm{kin}}(\boldsymbol{x}, \omega)$$

Extension of the Inertial Range

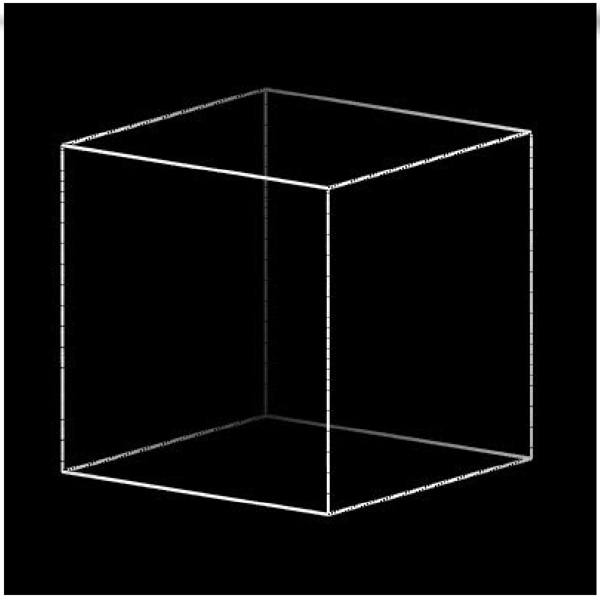




Inertial range becomes broad for time correlation.

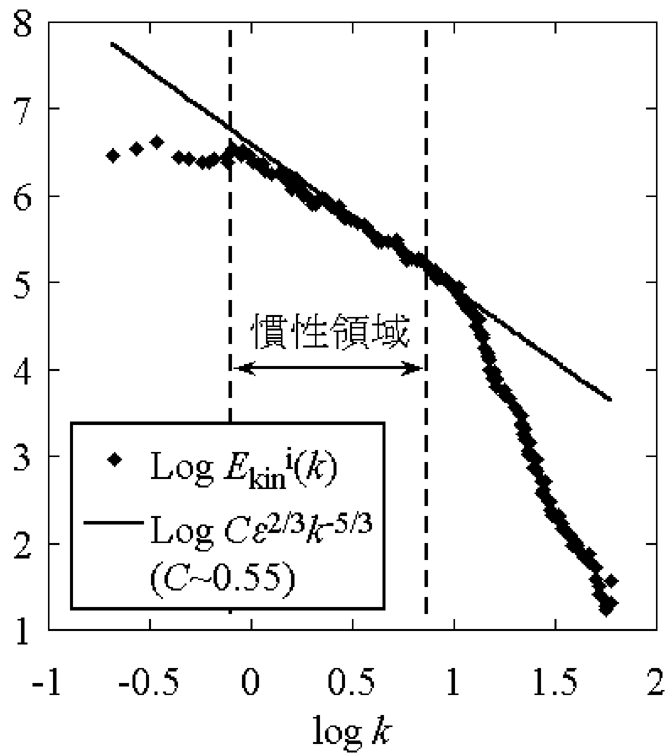
Extension of the Inertial Range

Injection of large vortex rings



Extension of the Inertial Range

256^3 grid



128^3 grid