Topological Excitations and Dynamical Behavior in Bose-Einstein Condensates and Other Systems

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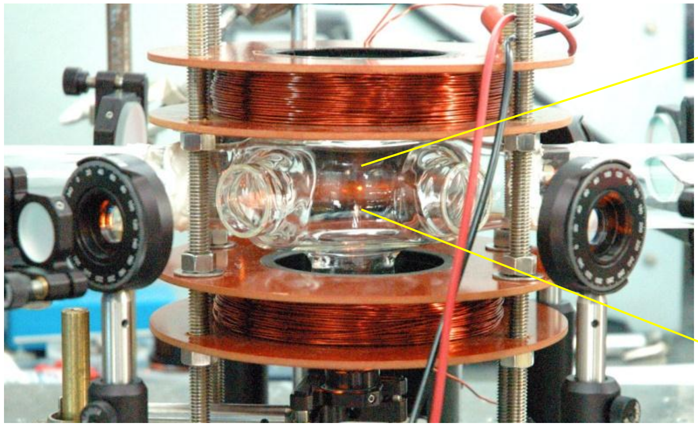
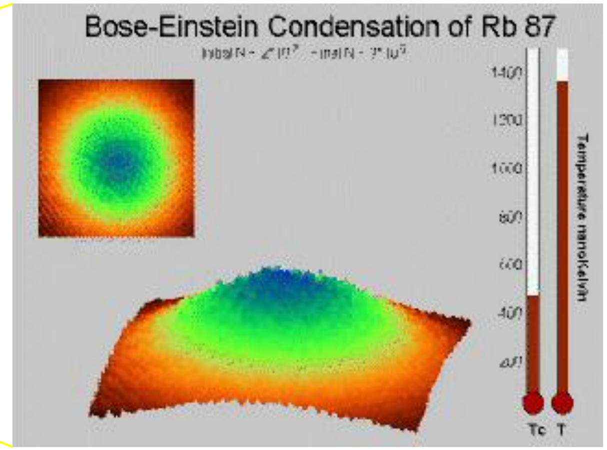




Scalar BEC without internal degrees of freedom

$$\rho(\boldsymbol{x},\boldsymbol{y}) = \langle \hat{\psi}(\boldsymbol{x})\hat{\psi}^{\dagger}(\boldsymbol{y})\rangle \overset{|\boldsymbol{x}-\boldsymbol{y}|\to\infty}{\longrightarrow} \psi(\boldsymbol{x})\psi(\boldsymbol{y})^{*}$$

$\psi(\boldsymbol{x}) = |\psi(\boldsymbol{x})| \exp[i\varphi(\boldsymbol{x})]$: broken $U(1)$ symmetry of global phase shift

Bose-Einstein Condensate with Internal Degrees of Freedom

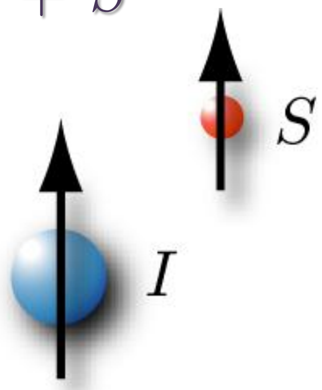
BEC with internal degrees of freedom

1. Multi-component BEC (ex. ^{87}Rb and ^{41}K BECs or different hyperfine level)
2. Spinor BEC (ex. ^{87}Rb → spin-1 and spin-2 BECs)

magnetic trap : spin degrees of freedom is frozen → scalar BEC
laser trap : spin degrees of freedom is alive → spinor BEC

Hyperfine spin : $F = I + S$

I : nuclear spin
S : electron spin



^{87}Rb, ^{23}Na, ^{7}Li, ^{41}K	$F=1,\ 2$
^{85}Rb	$F=2,\ 3$
^{133}Cs	$F=3,\ 4$
^{52}Cr	$F=3$

Symmetry and Topological Excitation in BEC

	Order parameter manifold (G/H)	Topological excitation
Scalar BEC	$U(1)/1 \simeq U(1)$	vortex
2-component BEC (miscible)	$\big(U(1) \times U(1)\big)/\mathbb{Z}_2$	vortex
2-component BEC (inmiscible)	$\big(U(1) \times U(1)\big)/(U(1)/\mathbb{Z}_2)$ $\simeq O(2) \simeq U(1) \rtimes \mathbb{Z}_2$	vortex & domain wall
Spin-1 BEC (ferro)	$(U(1) \times SO(3))/U(1) \simeq SO(3)$	vortex
Spin-1 BEC (polar)	$(U(1) \times SO(3))/(U(1) \rtimes \mathbb{Z}_2)$ $\simeq (U(1) \times S^2)/\mathbb{Z}_2$	vortex & monopole
Spin-2 BEC (ferro)	$(U(1) \times SO(3))/(U(1) \times \mathbb{Z}_2)$ $\simeq SO(3)/\mathbb{Z}_2$	vortex
Spin-2 BEC (uniaxial nematic)	$U(1) \times SO(3)/(U(1) \rtimes \mathbb{Z}_2)$ $\simeq U(1) \times \mathbb{RP}^2$	vortex & monopole
Spin-2 BEC (biaxial nematic)	$(U(1) \times SO(3))/D_4$	vortex (non-Abelian)
Spin-2 BEC (cyclic)	$(U(1) \times SO(3))/T$	vortex (non-Abelian)

Symmetry and Topological Excitation in BEC

	Order parameter manifold (G/H)	Topological excitation
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Theory of Spinor BEC

Hamiltonian of Bose system with spin

$$H = \int d\boldsymbol{x}_1 \frac{\hbar^2}{2M} \nabla\Psi_m^\dagger(\boldsymbol{x}_1)\nabla\Psi_m(x_1)$$

$$+\frac{1}{2}\int d\boldsymbol{x}_2 \Psi_{m_1}^\dagger(\boldsymbol{x}_1)\Psi_{m_2}^\dagger(\boldsymbol{x}_2)\, V_{m_1 m_2 m_1{}' m_2{}'}(\boldsymbol{x}_1-\boldsymbol{x}_2)\Psi_{m_2{}'}(\boldsymbol{x}_2)\Psi_{m_1{}'}(\boldsymbol{x}_1)$$

Low energy contact interaction $(l = 0)$

$$V_{m_1 m_1 m_1{}' m_2{}'}(\boldsymbol{x}_1-\boldsymbol{x}_2) = \delta(\boldsymbol{x}_1-\boldsymbol{x}_2) \sum_{F=\mathrm{even}} g_F \sum_{m_1 m_2 m_1{}' m_2{}' M} O^{FM}_{m_1 m_2}\left(O^{FM}_{m_1' m_2'}\right)^*$$

Coupling constant depends on total spin of two colliding particles

Theory of Spinor BEC

For spin-2 case

$$H = \int d\boldsymbol{x} \left[\frac{\hbar^2}{2M} \nabla\Psi_m^\dagger \nabla\Psi_m + \frac{c_0}{2} :n^2: + \frac{c_1}{2} :\boldsymbol{F}^2: + \frac{c_2}{2} A_{20}^{2}{}^\dagger A_{20}^2 \right]$$

$$c_0 = \frac{4g_2 + 3g_4}{7}, \qquad c_0 = \frac{g_4 - g_2}{7}, \qquad c_0 = \frac{7g_0 - 10g_2 + 3g_4}{35}$$

$n = \Psi_m^\dagger \Psi_m$: number density operator

$\boldsymbol{F} = \Psi_m^\dagger \boldsymbol{F}_{mn} \Psi_n$: spin density operator

$A_{20} = (-1)^m \Psi_m \Psi_{-m}$: time reversal operator (singlet-pair amplitude)

Theory of Spinor BEC

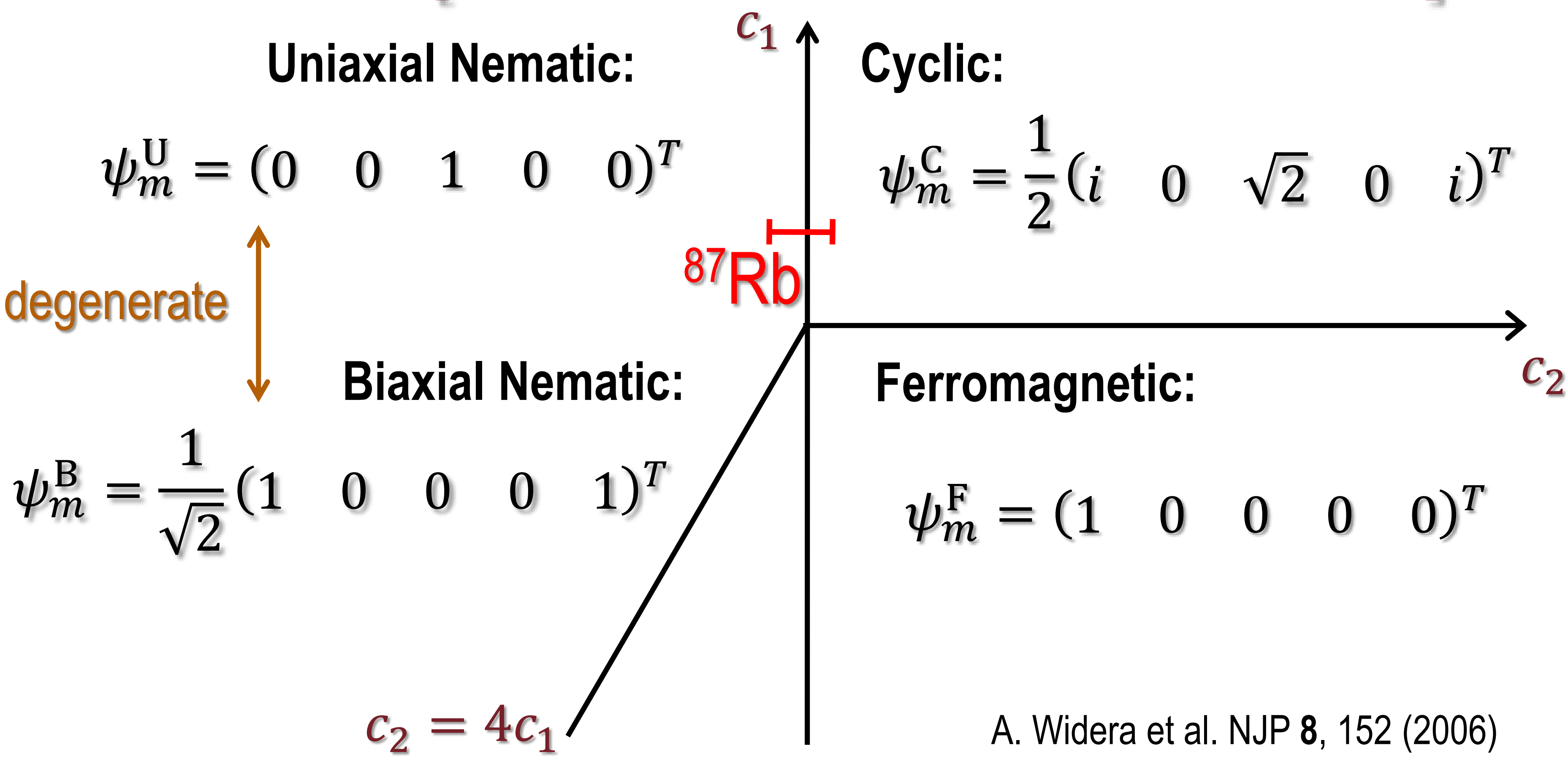
Mean-field theory at $T = 0$: $|\psi\rangle = \left(\psi_m a_{m,\boldsymbol{k}=0}\right)^N |0\rangle$
: all particles condense into a single-particle ground state

$$\langle H\rangle = \int d\boldsymbol{x} \left[\frac{\hbar^2}{2M}\nabla\psi_m^\dagger \nabla\psi_m + \frac{c_0}{2}n^2 + \frac{c_1}{2}\boldsymbol{F}^2 + \frac{c_2}{2}{A_{20}^2}^\dagger A_{20}^2\right]$$

$n = \psi_m^\dagger \psi_m$: number density
$\boldsymbol{F} = \psi_m^\dagger \boldsymbol{F}_{mn}\psi$: spin density
$A_{20} = (-1)^m \psi_m \psi_{-m}$: singlet-pair amplitude

Phase Diagram for Ground State

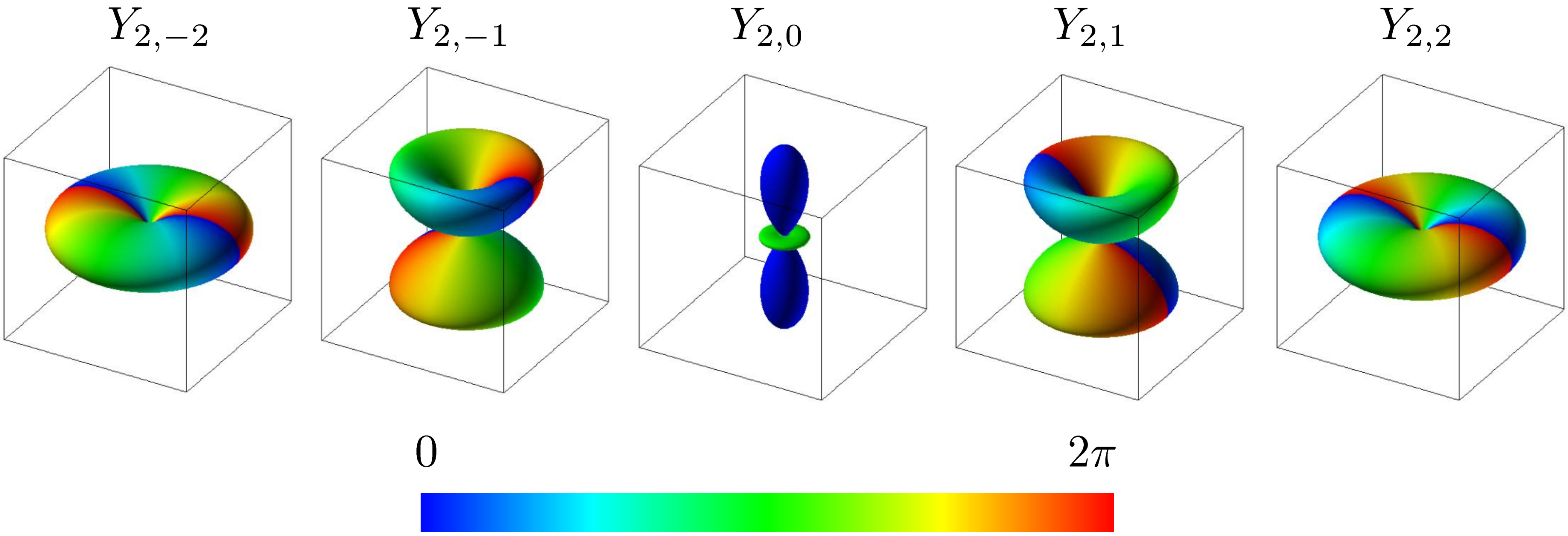
$$\langle H \rangle = \int d\boldsymbol{x} \left[\frac{\hbar^2}{2M} \nabla\psi_m^\dagger \nabla\psi_m + \frac{c_0}{2} n^2 + \frac{c_1}{2} \boldsymbol{F}^2 + \frac{c_2}{2} {A_{20}^2}^\dagger A_{20}^2 \right]$$



A. Widera et al. NJP **8**, 152 (2006)

Spherical Harmonic Representation

$$\psi(\theta,\phi) = \sum_{m=-2}^{2} \psi_m Y_{2,m}(\theta,\phi)$$

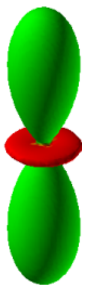


Phase Diagram for Ground State

$$\langle H \rangle = \int d\boldsymbol{x} \left[\frac{\hbar^2}{2M} \nabla \psi_m^\dagger \nabla \psi_m + \frac{c_0}{2} n^2 + \frac{c_1}{2} \boldsymbol{F}^2 + \frac{c_2}{2} {A_{20}^2}^\dagger A_{20}^2 \right]$$

c_1

Uniaxial Nematic:



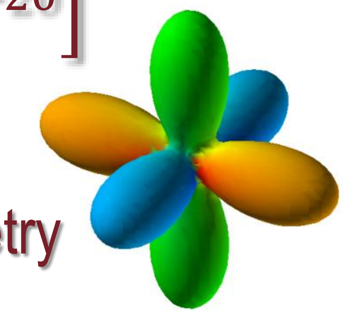
$$\psi_m^{\mathrm{U}} = (0 \quad 0 \quad 1 \quad 0 \quad 0)^T$$

D_∞ : cylindrical symmetry

^{87}Rb

Cyclic:

T : tetrahedral symmetry



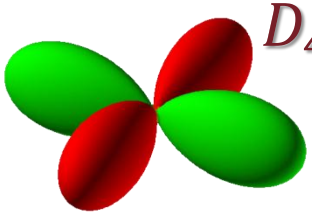
$$\psi_m^{\mathrm{C}} = \frac{1}{2}(i \quad 0 \quad \sqrt{2} \quad 0 \quad i)^T$$

c_2

Biaxial Nematic:

$$\psi_m^{\mathrm{B}} = \frac{1}{\sqrt{2}}(1 \quad 0 \quad 0 \quad 0 \quad 1)^T$$

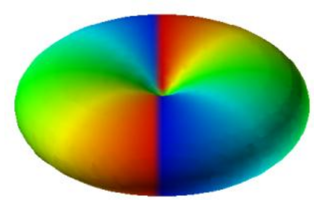
D_4 : square symmetry



$c_2 = 4c_1$

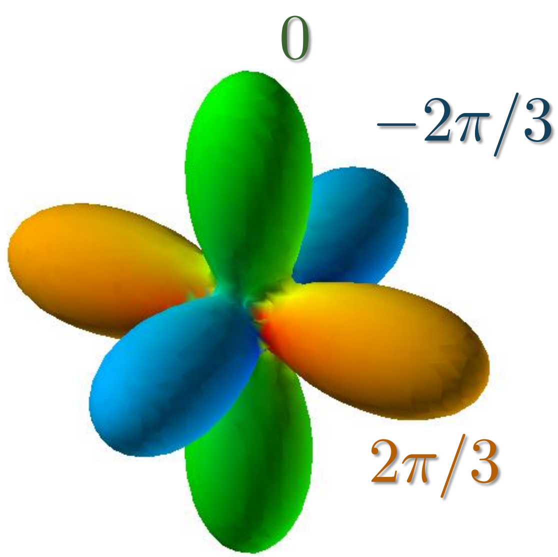
Ferromagnetic:

$$\psi_m^{\mathrm{F}} = (1 \quad 0 \quad 0 \quad 0 \quad 0)^T$$

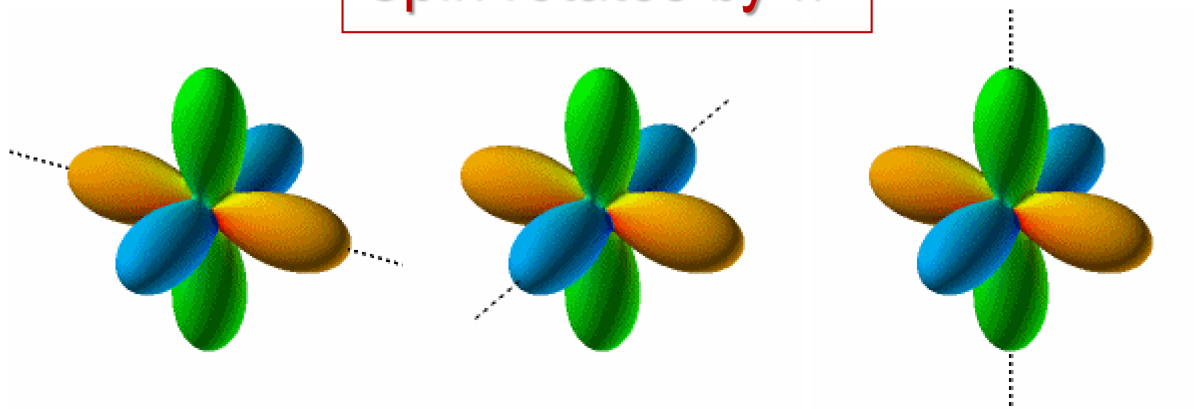


$U(1) \times \mathbb{Z}_2$: oriented toroidal symmetry

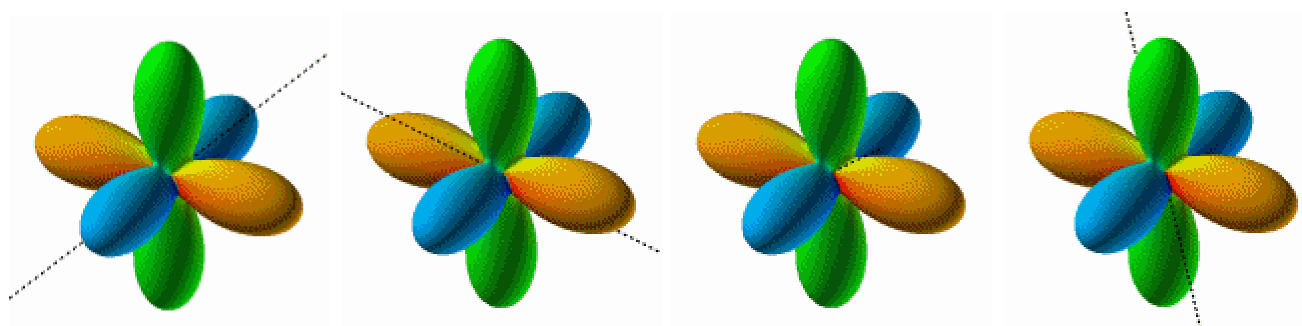
Symmetry of cyclic state



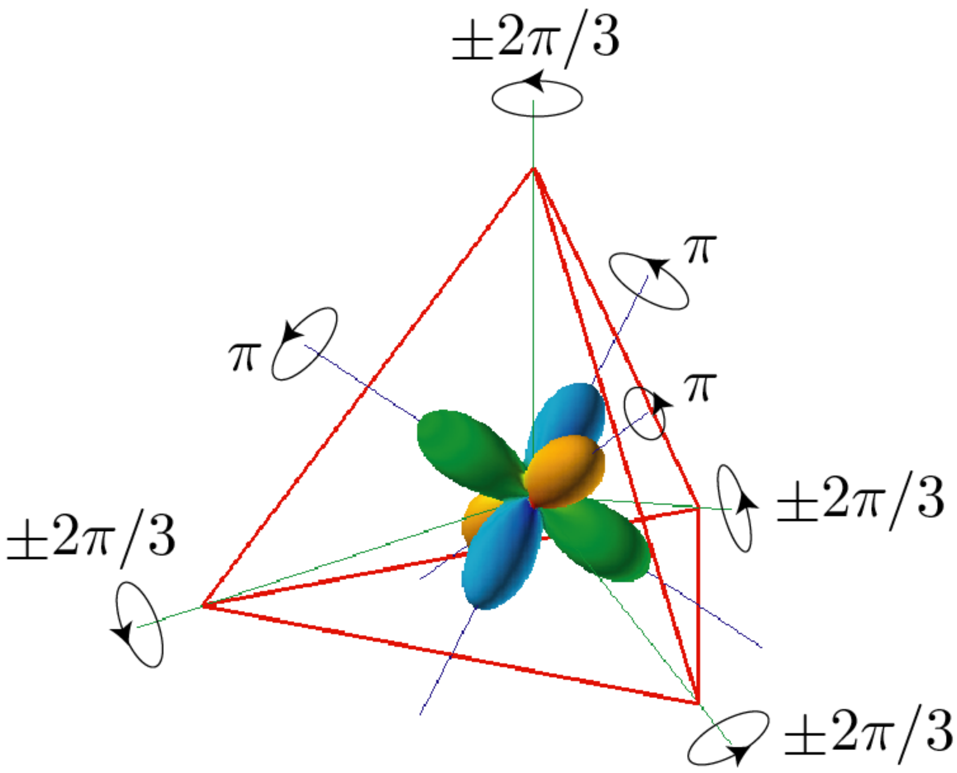
Spin rotates by π



Phase shift by $\pm 2\pi/3$ and spin rotates by $\pm\, 2\pi/3$



Symmetry of cyclic state



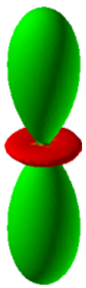
Spin rotations keeping cyclic state invariant form a non Abelian tetrahedral symmetry

Phase Diagram for Ground State

$$\langle H\rangle = \int d\boldsymbol{x}\left[\frac{\hbar^2}{2M}\nabla\psi_m^\dagger\nabla\psi_m + \frac{c_0}{2}n^2 + \frac{c_1}{2}\boldsymbol{F}^2 + \frac{c_2}{2}{A_{20}^2}^\dagger A_{20}^2\right]$$

c_1

Uniaxial Nematic:

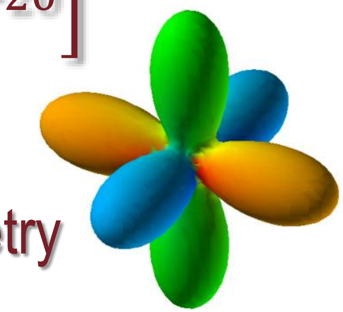


$$\psi_m^{\mathrm{U}} = (0 \quad 0 \quad 1 \quad 0 \quad 0)^T$$

D_∞ : cylindrical symmetry

Cyclic:

T : tetrahedral symmetry



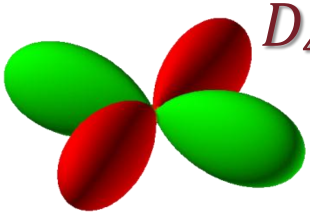
$$\psi_m^{\mathrm{C}} = \frac{1}{2}(i \quad 0 \quad \sqrt{2} \quad 0 \quad i)^T$$

c_2

Biaxial Nematic:

$$\psi_m^{\mathrm{B}} = \frac{1}{\sqrt{2}}(1 \quad 0 \quad 0 \quad 0 \quad 1)^T$$

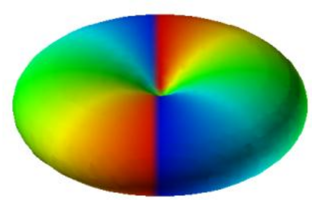
D_4 : square symmetry



$c_2 = 4c_1$

Ferromagnetic:

$$\psi_m^{\mathrm{F}} = (1 \quad 0 \quad 0 \quad 0 \quad 0)^T$$



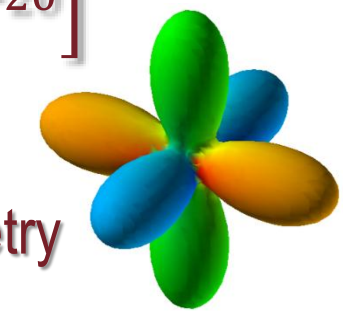
$U(1)\times\mathbb{Z}_2$: oriented toroidal symmetry

Phase Diagram for Ground State

$$\langle H \rangle = \int d\boldsymbol{x} \left[\frac{\hbar^2}{2M} \nabla \psi_m^\dagger \nabla \psi_m + \frac{c_0}{2} n^2 + \frac{c_1}{2} \boldsymbol{F}^2 + \frac{c_2}{2} {A_{20}^2}^\dagger A_{20}^2 \right]$$

c_1

Cyclic:



T : tetrahedral symmetry

$$\psi_m^{\mathrm{C}} = \frac{1}{2} \begin{pmatrix} i & 0 & \sqrt{2} & 0 & i \end{pmatrix}^T$$

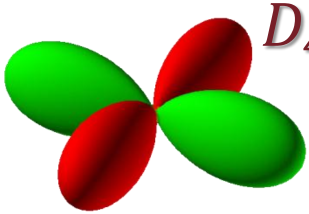
Non-Abelian vortices appear due to non-Abelian discrete symmetry

c_2

Biaxial Nematic:

$$\psi_m^{\mathrm{B}} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 0 & 1 \end{pmatrix}^T$$

D_4 : square symmetry



$c_2 = 4c_1$

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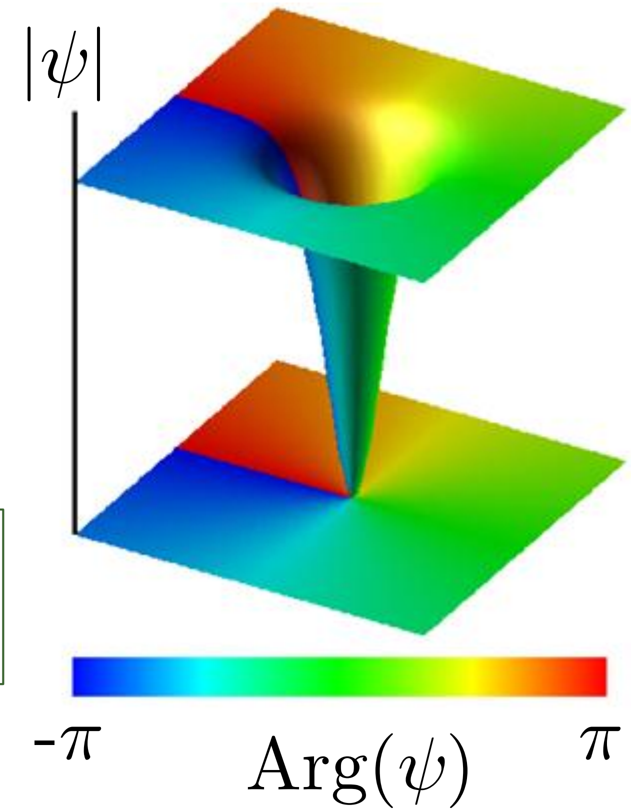
Quantized Vortices in BEC

Quantized vortex for $m = +1$



For scalar BEC : $\psi = |\psi| e^{im\varphi}$

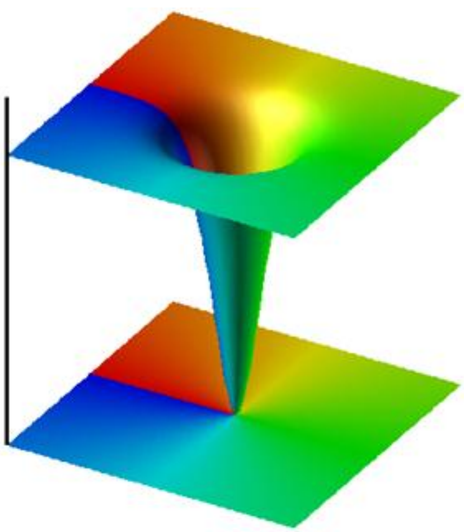
Topological charge can be characterized by widing number m (additive group of integers)

Non-Abelian Vortex

Topological charge of vortices

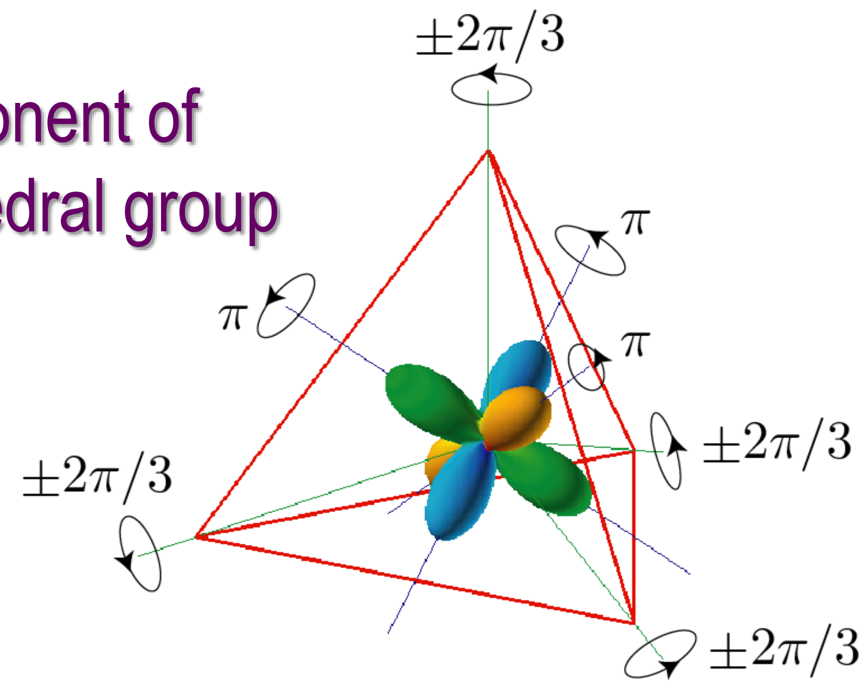
Scalar BEC

Integer (winding of phase by 2π multiple)



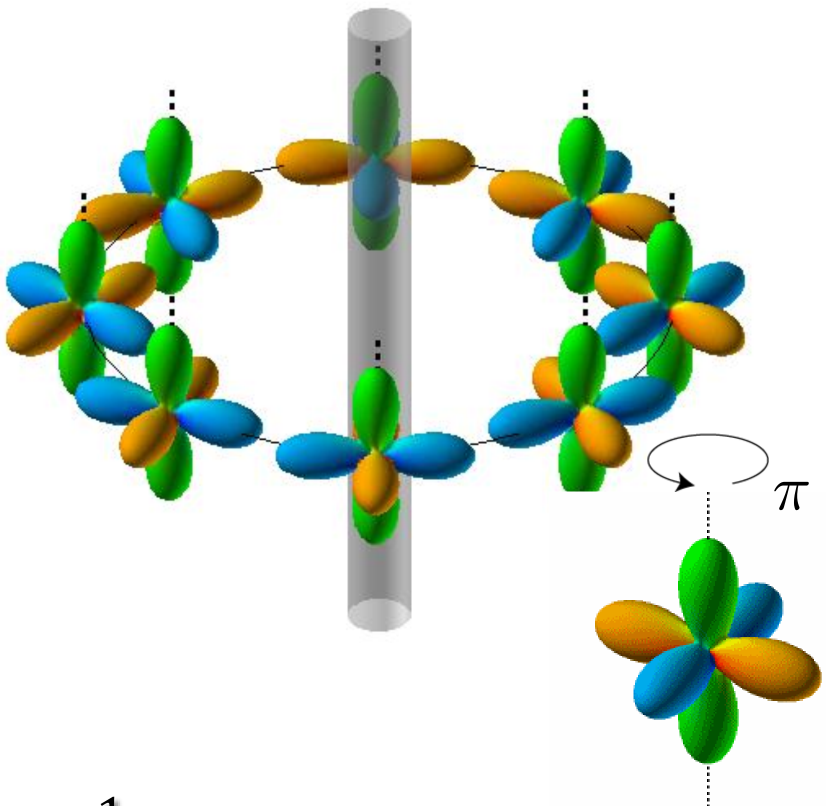
Cyclic phase in spin-2 spinor BEC

Component of tetrahedral group



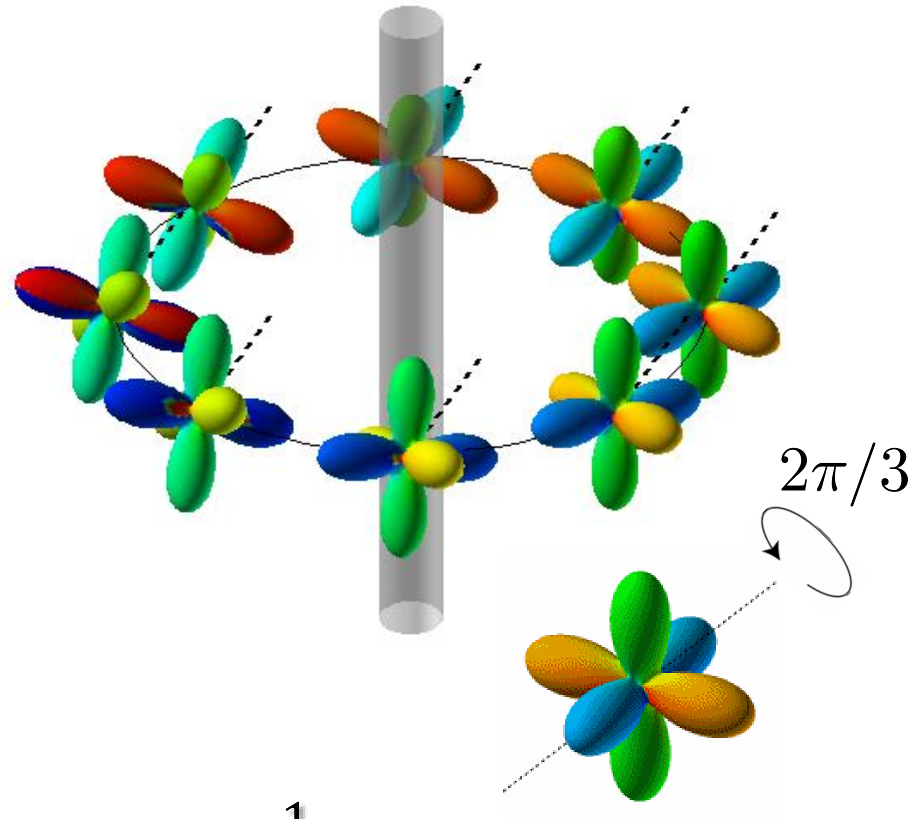
Vortices in cyclic state

1/2 spin vortex



$$\psi = \frac{1}{2}\left(ie^{i\varphi} \quad 0 \quad \sqrt{2} \quad 0 \quad ie^{-i\varphi}\right)^T$$

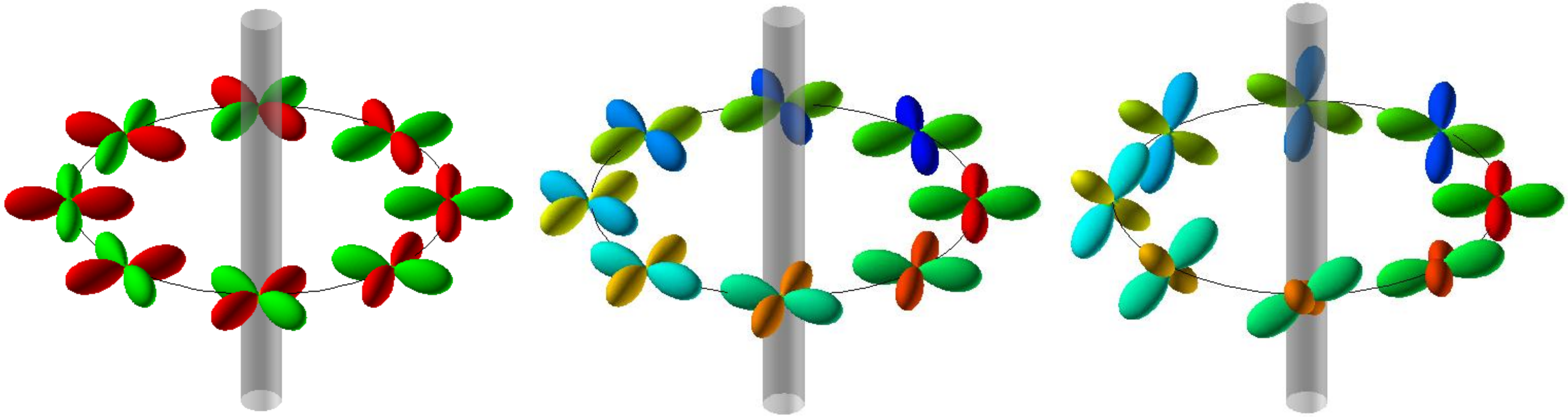
1/3 vortex



$$\psi = \frac{1}{\sqrt{3}}\left(e^{i\varphi} \quad 0 \quad 0 \quad \sqrt{2} \quad 0\right)^T$$

Vortices in biaxial nematic state

1/2 spin vortex — 1/4 vortex — 1/2 vortex



$$\frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\varphi} & 0 & 0 & 0 & e^{-i\varphi} \end{pmatrix}^T \quad \frac{1}{\sqrt{2}}\begin{pmatrix} e^{i\varphi} & 0 & 0 & 0 & 1 \end{pmatrix}^T \quad \frac{1}{\sqrt{2}}\begin{pmatrix} 0 & e^{i\varphi} & 0 & 1 & 0 \end{pmatrix}^T$$

Contents

Gross-Pitaevskii Equation

Coherent dynamics of mean-field : Gross-Pitaevskii equation

$$i\hbar\frac{\partial\psi_m}{\partial t} = \frac{\delta H}{\delta\psi_m^*}$$

$$i\hbar\frac{\partial\psi_{\pm2}}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\psi_{\pm2} + c_0 n\psi_{\pm2} + c_1(F_{\mp}\psi_1 \pm 2F_z\psi_{\pm2}) + \frac{c_2}{\sqrt{5}}A_{00}\psi_{\mp2}^*$$

$$i\hbar\frac{\partial\psi_{\pm1}}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\psi_{\pm1} + c_0 n\psi_{\pm1} + c_1\left(\frac{\sqrt{6}}{2}F_{\mp0}\psi_0 + F_{\pm}\psi_{\pm2} \pm F_z\psi_{\pm1}\right) - \frac{c_2}{\sqrt{5}}A_{00}\psi_{\mp1}^*$$

$$i\hbar\frac{\partial\psi_0}{\partial t} = -\frac{\hbar^2}{2M}\nabla^2\psi_0 + c_0 n\psi_0 + \frac{\sqrt{6}}{2}c_1(F_-\psi_{-1} + F_+) + \frac{c_2}{\sqrt{5}}A_{00}\psi_0^*$$

Collision Dynamics

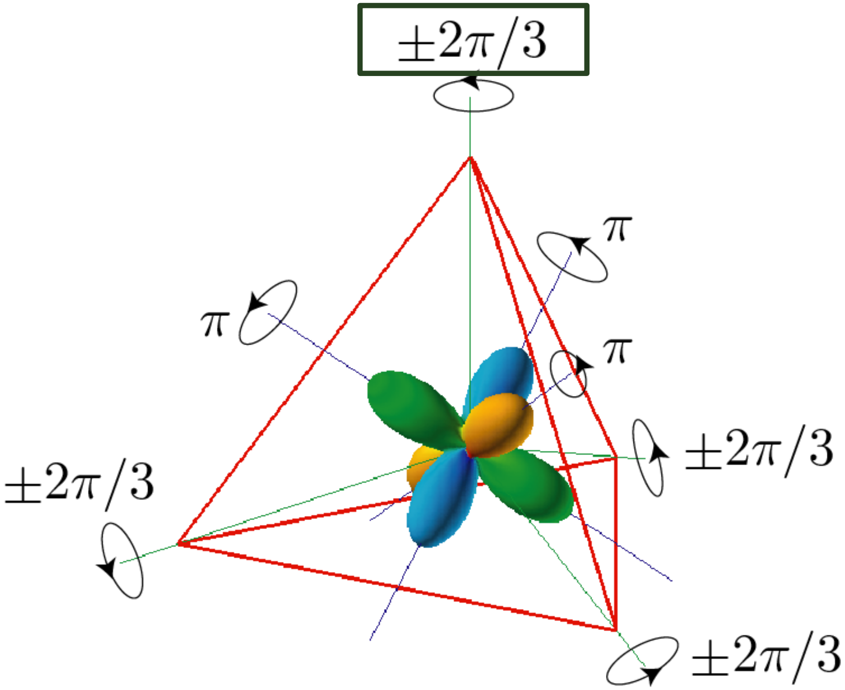
For Abelian vortex

Reconnect
: All Abelian vortices such as those in scalar BEC

Pass through
: Rarely seen for quantized vortices, and sometimes seen for disclination in liquid crystals or cosmic strings

Collision Dynamics for Cyclic Phase

There are 12 kinds of vortices in cyclic phase

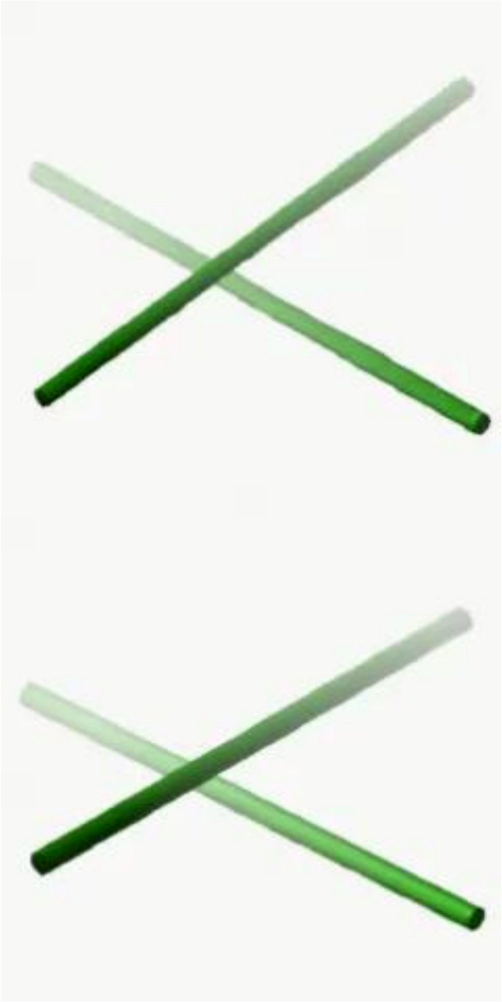


For same kinds of vortices
: $+2\pi/3$ & $+2\pi/3$ →reconnection

For different and commutative vortices
: $+2\pi/3$ & $-2\pi/3$ →pass through

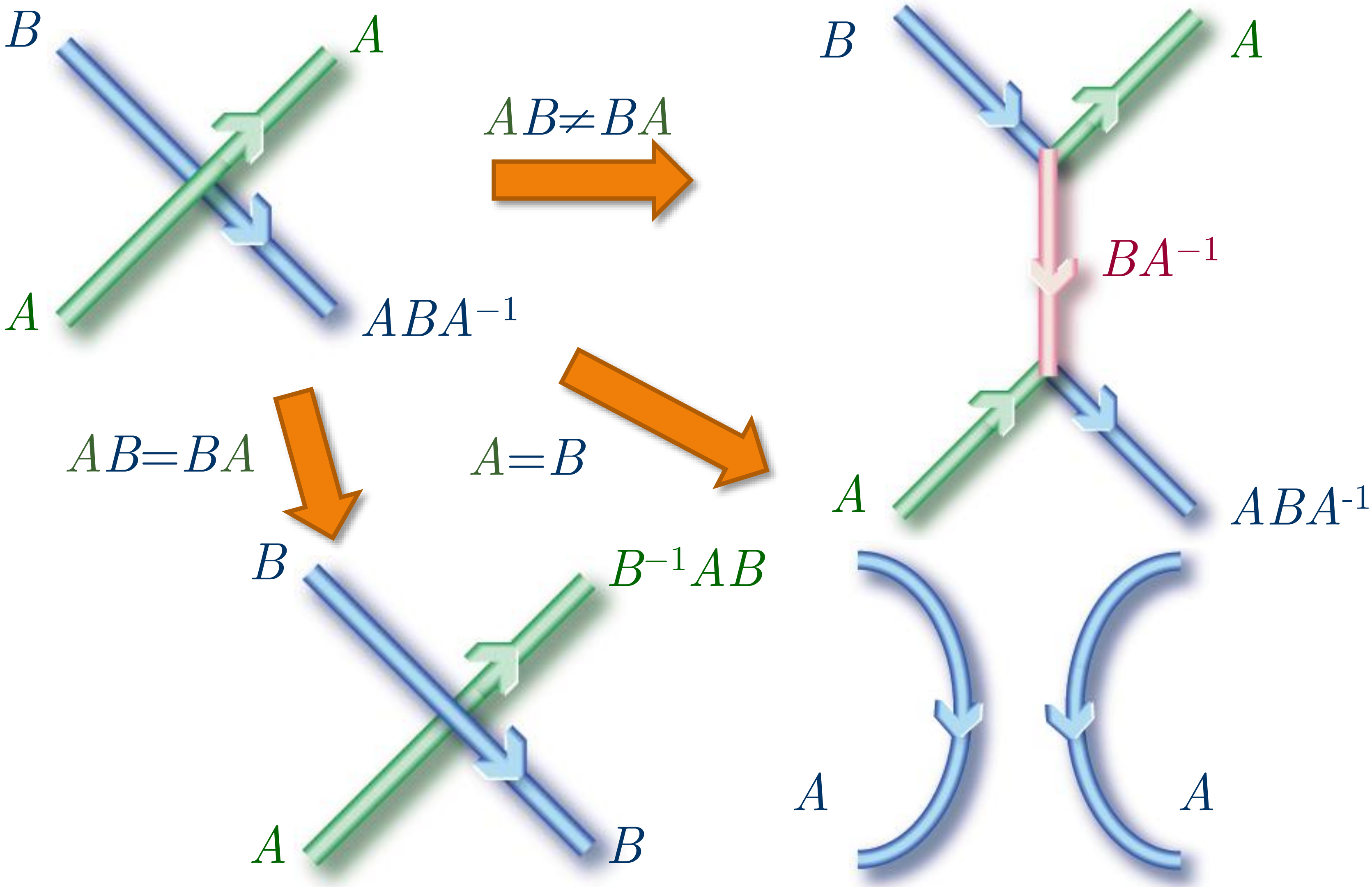
What happens for non-commutative vortices for different spin rotations?

Collision Dynamics for Cyclic Phase



New “rung” vortex appears bridging two colliding vortices

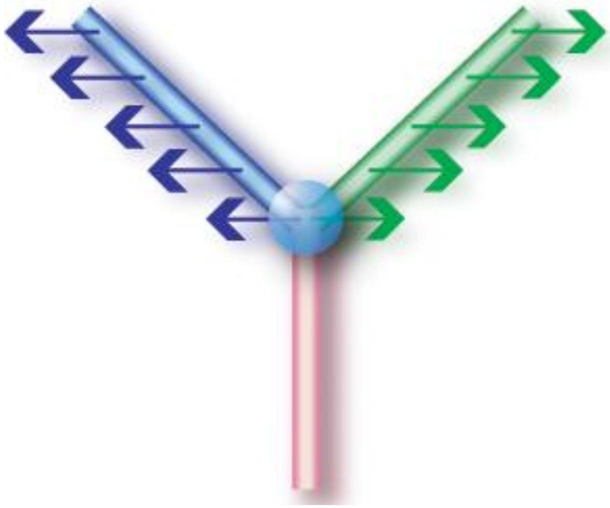
Collision of Vortex



Monopole Confined in Vortex Junction

Vortex core has usually internal structure different from cyclic state depending on the charge of vortex (ex. ferromagnetic state with $\boldsymbol{F} \neq 0$)

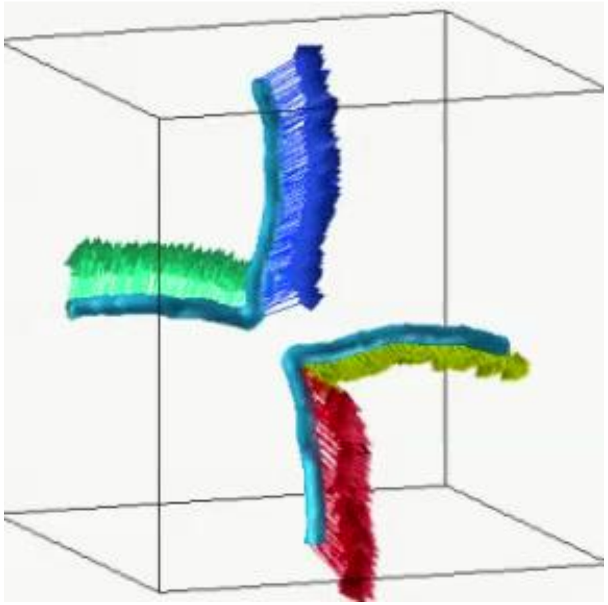
Monopole ($\mathrm{div}\boldsymbol{F} \neq 0$) appears at the Y-shape junction point



Magnetization and its divergence is confined to only vortex lines

→ **confined monopole** (charge is classified by the tetrahedral symmetry)

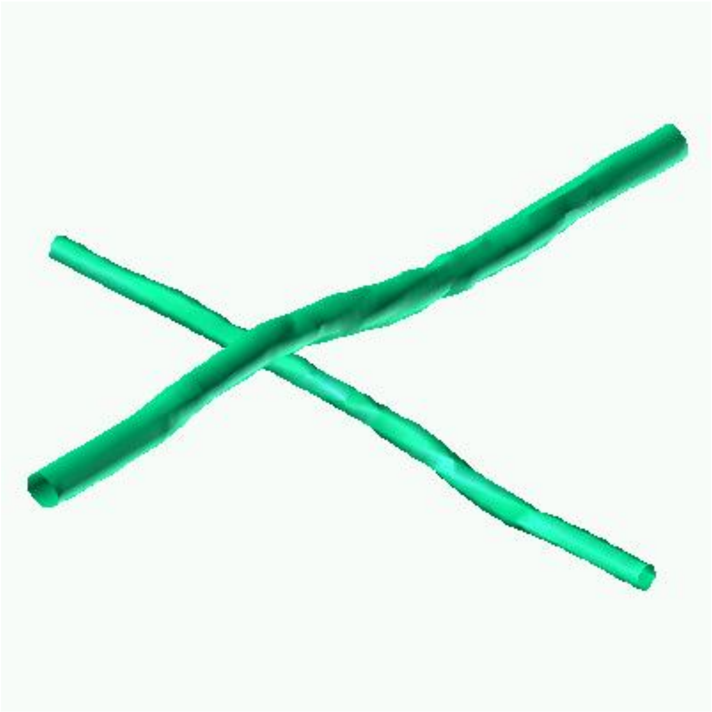
Monopole Confined in Vortex Junction



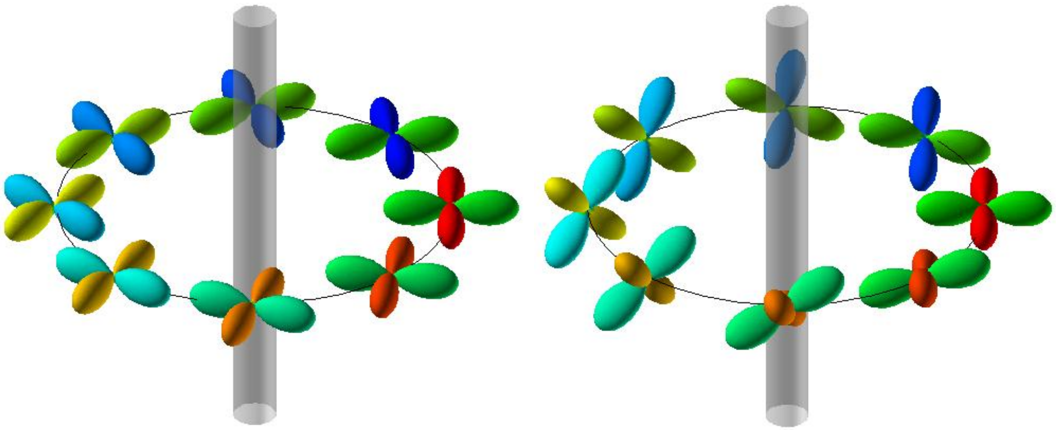
•Each arrow shows the direction of magnetization

Confined monopoles appear at the junctions points of vortices as a form of monopole-antimonopole pair

Collision Dynamics in Biaxial Nematic Phase



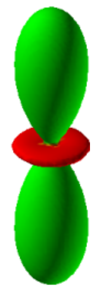
1/4 vortex 1/2 vortex



Rung vortex burst and disappears
→Non-Abelian property cannot be seen

Degeneration between Uniaxial and Biaxial Nematic Phases

$$\langle H\rangle = \int dx \left[\frac{\hbar^2}{2M}\nabla\psi_m^\dagger\nabla\psi_m + \frac{c_0}{2}n^2 + \frac{c_1}{2}\boldsymbol{F}^2 + \frac{c_2}{2}{A_{20}^2}^\dagger A_{20}^2\right]$$



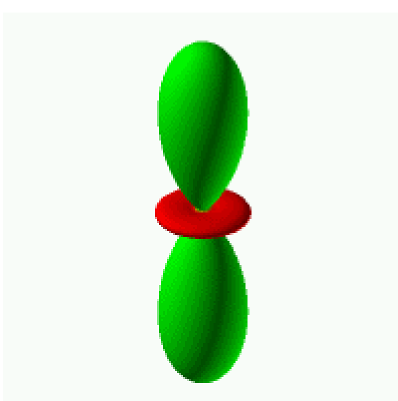
Uniaxial Nematic:

$$\psi_m^{\mathrm{U}} = (0 \quad 0 \quad 1 \quad 0 \quad 0)^T$$

D_∞ : cylindrical symmetry

degeneration with another continuous degree of freedom

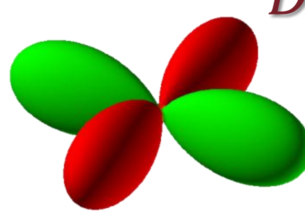
$$\frac{1}{\sqrt{2}}(\cos\eta \quad 0 \quad \sqrt{2}\sin\eta \quad 0 \quad \cos\eta)^T$$

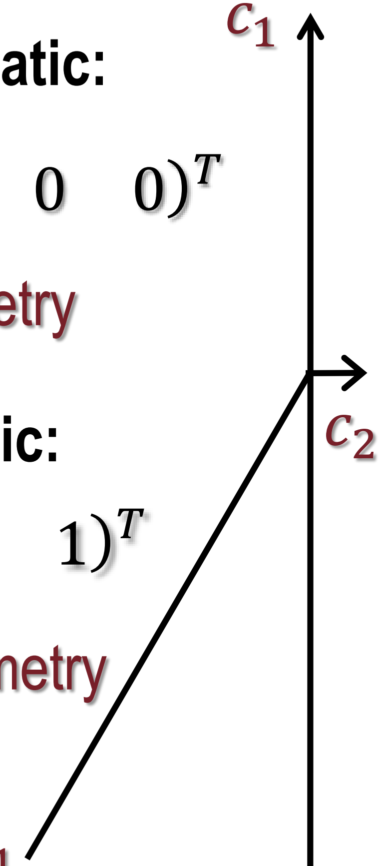


Biaxial Nematic:

$$\psi_m^{\mathrm{B}} = \frac{1}{\sqrt{2}}(1 \quad 0 \quad 0 \quad 0 \quad 1)^T$$

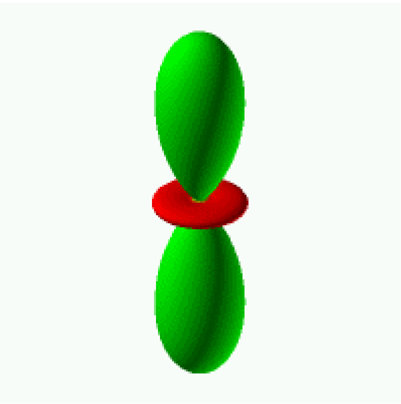
D_4 : square symmetry





Degeneration between Uniaxial and Biaxial Nematic Phases

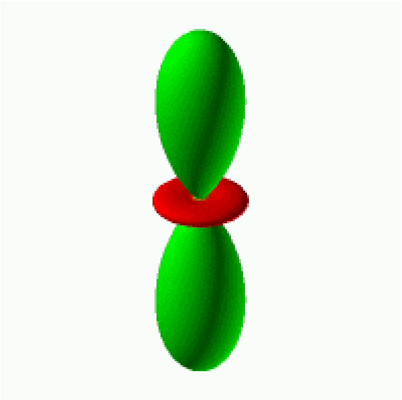
Uniaxial Nematic: **Biaxial Nematic:**



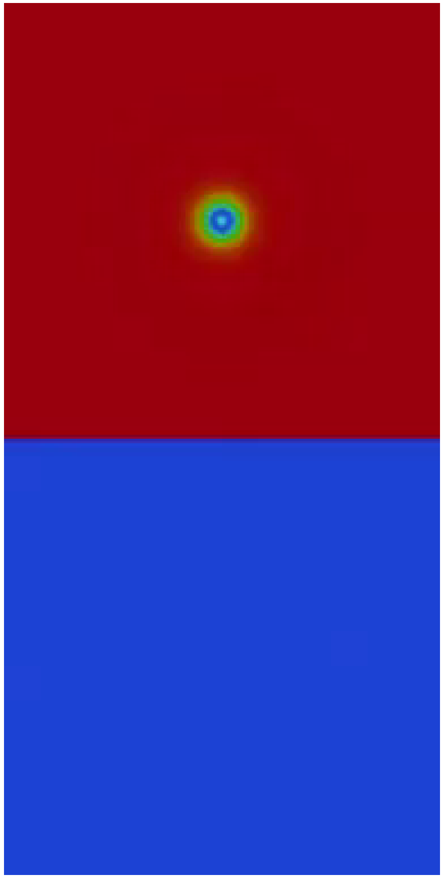
$$\frac{1}{\sqrt{2}}\begin{pmatrix}\cos\eta & 0 & \sqrt{2}\sin\eta & 0 & \cos\eta\end{pmatrix}^T$$

Very large order-parameter manifold
: $(U(1) \times S^4)/\mathbb{Z}_2$
→Several vortices in both phases are topologically unstable due to η

Quasi-Nambu-Goldstone Mode



$$\frac{1}{\sqrt{2}}\begin{pmatrix}\cos\eta & 0 & \sqrt{2}\sin\eta & 0 & \cos\eta\end{pmatrix}^T$$

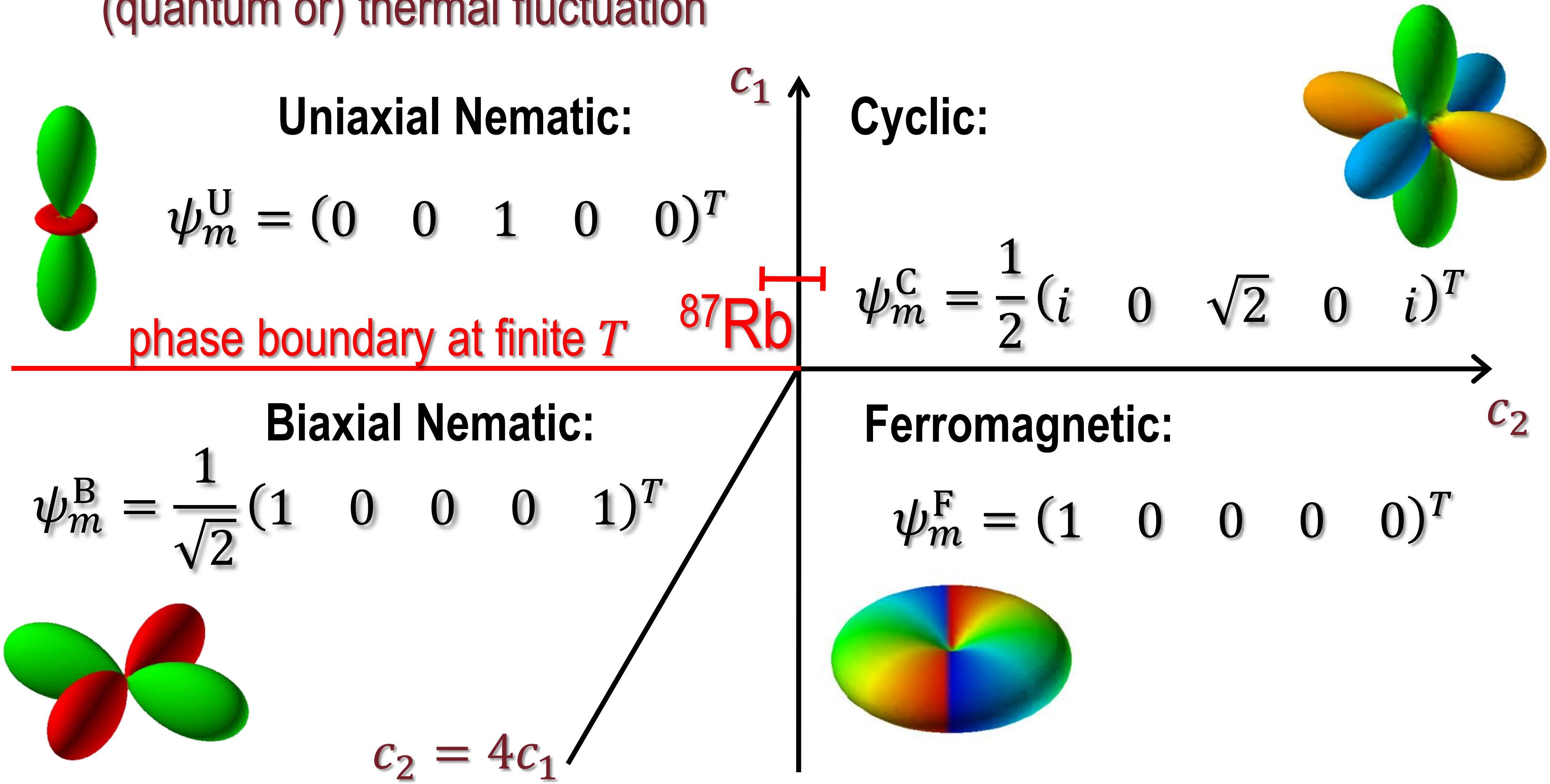


η is not the symmetry of the Hamiltonian (accidental symmetry)

→Gapless excitation mode due to η is not the true Nambu-Goldstone mode
 (Quasi-Nambu-Goldstone mode)

Thermal Phase Diagram

Quasi-Nambu-Goldstone mode easily becomes gapful through (quantum or) thermal fluctuation

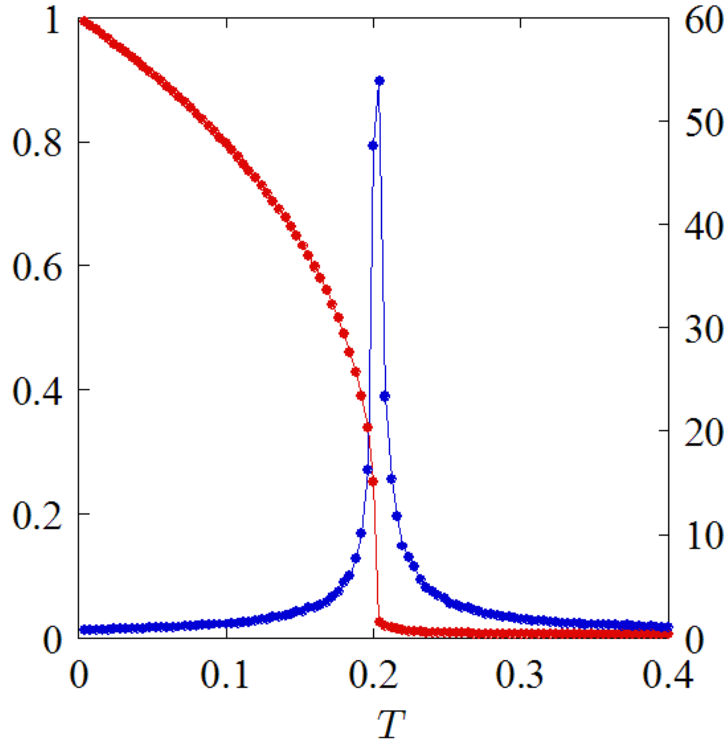


BEC at Finite Temperature

Stochastic Gross-Pitaevskii equation (complex Langevin equation)

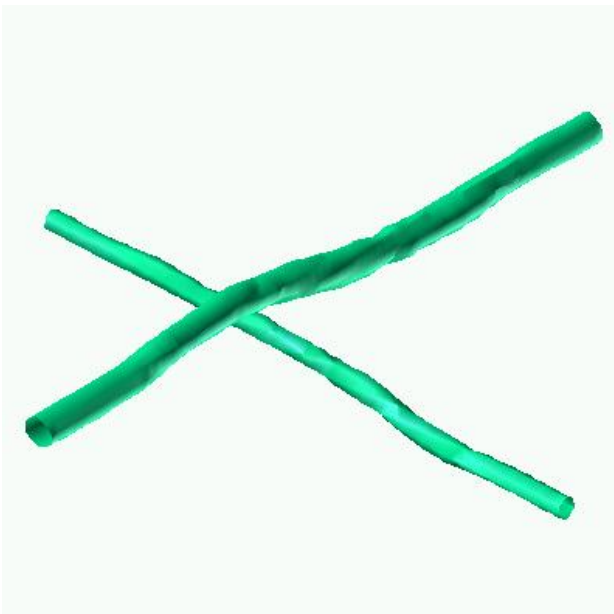
$$i\hbar \frac{\partial \psi_m}{\partial t} = (1-\gamma)\left(\frac{\delta H}{\delta \psi_m}\right) + \xi \qquad \langle \xi(\boldsymbol{x}, t)\xi(\boldsymbol{x}', t')\rangle = k_{\mathrm{B}} T \delta(x - x')\delta(t - t')$$

Condensate fraction and its fluctuation

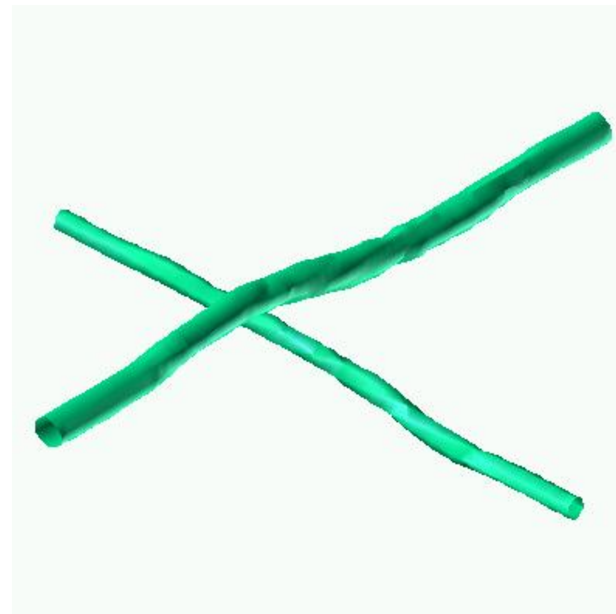


Collision Dynamics at Finite Temperature

$T = 0.1T_c$



$T = 0.4T_c$



Non-Abelian property is restored by massive quasi-Nambu-Goldstone mode due to thermal fluctuation

Summary

1. Spin-2 spinor BEC can have exotic non-Abelian vortex due to non-Abelian discrete symmetry.
2. Collision of non-Abelian vortex in the cyclic phase show the formation of rung vortex bridging colliding vortices.
3. After the collision, there appears a monopole confined in the Y-shaped junction.
4. Non-Abelian property disappears in the biaxial nematic phase due to the quasi-Nambu-Goldstone mode, and is restored by thermal fluctuations at finite temperatures.